RECONSTRUCTION OF INERTIA GROUPS ASSOCIATED TO LOG DIVISORS FROM A CONFIGURATION SPACE GROUP EQUIPPED WITH ITS COLLECTION OF LOG-FULL SUBGROUPS

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Abstract. In the present paper, we study configuration space groups. The goal of this paper is to reconstruct group-theoretically the inertia groups associated to various types of log divisors of a log configuration space of a smooth log curve from the associated configuration space group equipped with its collection of log-full subgroups.

0. **Introduction**

Let l be a prime number; k an algebraically closed field of characteristic $\neq l$; $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$; (g, r) a pair of nonnegative integers such that $2g-2+r > 0$; $X^{\log} \to S$ a smooth log curve of type (g, r) (cf. Notation 1.3, (iv)); $n \in \mathbb{Z}_{>1}$. In the present paper, we study the n-th log configuration space $X_n^{\log}$ associated to $X^{\log} \to S$ (cf. Definition 2.1). Write U_X for the interior of the log scheme $X^{\log}$ (cf. Notation 1.2, (vi)). The log scheme $X_n^{\log}$ may be thought of as a certain compactification of the usual n-th configuration space U_{X_n} associated to the smooth curve U_X. Write $\Pi_n \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l}(X_n^{\log})$ for the pro-l configuration space group determined by $X_n^{\log}$ (cf. [MzTa], Definition 2.3, (i)), i.e., the maximal pro-l quotient of the fundamental group of the log scheme $X_n^{\log}$ (for a suitable choice of basepoint). We shall refer to an irreducible divisor of the underlying scheme of $X_n^{\log}$ contained in the complement of U_{X_n} as a *log divisor* of $X_n^{\log}$. Each log divisor V determines, up to Π_n-conjugacy, an inertia group $I_V (\simeq \mathbb{Z}_l) \subseteq \Pi_n$, which plays a central role in the present paper. Let $V_1, \ldots, V_n$ be distinct log divisors of $X_n^{\log}$ such that $V_1 \cap \cdots \cap V_n \neq \emptyset$. Then we shall refer to $P \stackrel{\text{def}}{=} V_1 \cap \cdots \cap V_n$ as a *log-full point* (cf. Definition 2.2, (ii), and Proposition 2.10). The log-full point $P = V_1 \cap \cdots \cap V_n$ determines, up to Π_n-conjugacy, a *log-full subgroup* A $(\simeq I_{V_1} \times \cdots \times I_{V_n} \simeq \mathbb{Z}_l^{\oplus n}) \subseteq \Pi_n$ (cf. Definition 2.2, (iii)). It is known that the log-full subgroups of a configuration space group may be characterized group-theoretically whenever the configuration space group is equipped with the action of a profinite group that satisfies certain properties (cf. [HMM], Theorem D). In the present paper, we reconstruct group-theoretically the inertia groups associated to the log divisors from a configuration space group equipped with its collection of log-full subgroups. Moreover, we reconstruct group-theoretically the inertia groups associated to the tripodal divisors (cf. Definition 3.1, (ii)) and the drift diagonals (cf. Definition 3.1, (iv)), as well as the drift collections of Π_n (cf. Definition 8.13) and the generalized fiber subgroups of Π_n (cf. Definition 9.1).

Our main result is as follows:

Theorem 0.1. *For $\Box \in \{\circ, \bullet\}$, let $l^{\Box}$ be a prime number; $k^{\Box}$ an algebraically closed field of characteristic $\neq l^{\Box}$; $S^{\Box} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^{\Box})$; $(g^{\Box}, r^{\Box})$ a pair of nonnegative integers such that $2g^{\Box} - 2 + r^{\Box} > 0$;*

$$X^{\log \Box} \to S^{\Box}$$

(cf. Notation 1.2, (vi)) a smooth log curve of type $(g^{\Box}, r^{\Box})$; $n^{\Box} \in \mathbb{Z}_{>1}$; $X^{\log \Box}_{n^{\Box}}$ the $n^{\Box}$-th log configuration space associated to $X^{\log \Box} \to S^{\Box}$; $\Pi^{\Box} \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^{\Box}}(X^{\log \Box}_{n^{\Box}})$ (for a suitable choice of basepoint);

$$\phi \colon \Pi^{\circ} \xrightarrow{\sim} \Pi^{\bullet}$$

an isomorphism of profinite groups. We suppose that $r^{\Box} > 0$ (cf. the discussion below); ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^{\bullet}$. Then the following hold:

(i) ϕ induces a bijection between the set of inertia groups of Π° associated to log divisors of $X^{\log \circ}_{n^{\circ}}$ and the set of inertia groups of $\Pi^{\bullet}$ associated to log divisors of $X^{\log \bullet}_{n^{\bullet}}$ (cf. Theorem 5.2).

(ii) ϕ induces a bijection between the set of inertia groups of Π° associated to tripodal divisors of $X^{\log \circ}_{n^{\circ}}$ and the set of inertia groups of $\Pi^{\bullet}$ associated to tripodal divisors of $X^{\log \bullet}_{n^{\bullet}}$ (cf. Theorem 6.6).

(iii) ϕ induces a bijection between the set of inertia groups of Π° associated to drift diagonals of $X^{\log \circ}_{n^{\circ}}$ and the set of inertia groups of $\Pi^{\bullet}$ associated to drift diagonals of $X^{\log \bullet}_{n^{\bullet}}$ (cf. Theorem 7.3).

(iv) ϕ induces a bijection between the set of drift collections of Π° and the set of drift collections of $\Pi^{\bullet}$ (cf. Theorem 8.14).

(v) ϕ induces a bijection between the set of generalized fiber subgroups of Π° and the set of generalized fiber subgroups of $\Pi^{\bullet}$ (cf. Theorem 9.3).

Note that, roughly speaking, Theorem 0.1, (i), asserts that we may extract group-theoretically a "geometric direct summand $\mathbb{Z}_l$" (i.e., an inertia group associated to a log divisor) from "$\mathbb{Z}_l^{\oplus n}$" (i.e., a log-full subgroup).

Note that one may define the notion of a log-full point even if $r = 0$ (cf. [HMM], Definition 1.1). On the other hand, since log-full points do not exist when $r = 0$, we suppose that $r > 0$ in the present paper.

In the proof of Theorem 0.1, (ii), we use the fact that, in the notation of Theorem 0.1, in fact $(g^{\circ}, r^{\circ}, n^{\circ}) = (g^{\bullet}, r^{\bullet}, n^{\bullet})$ (cf. Theorem 3.10, (i)), which is proven in [HMM], Theorem A, (i).

This paper is organized as follows: In §1, we explain some notations. In §2, we define log configuration spaces, log-full points, and log divisors. In §3, we define tripodal divisors and drift diagonals and then proceed to study the geometry of various types of log divisors. In §4, we give a group-theoretic reconstruction of the scheme-theoretically non-degenerate elements (cf. Definition 4.6, (i)) of a log-full subgroup. In §5, we reconstruct the inertia groups associated to the log divisors. In §6, we reconstruct the inertia groups associated to the tripodal divisors. In §7, we reconstruct the inertia groups associated to the drift diagonals. In §8, we reconstruct the drift collections of a configuration space group. In §9, we reconstruct the generalized fiber subgroups of a configuration space group.

1. **Notations**

Notation 1.1. (i) Let G be a group. Then we shall write "$1_G \in G$" for the identity element of G.

(ii) Let G be a group, $H \subseteq G$ a subgroup, and $\alpha \in G$. Then we shall write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid gh = hg \text{ for any } h \in H\}$$

for the centralizer of H in G;

$$Z_G(\alpha) \stackrel{\text{def}}{=} Z_G(\langle\alpha\rangle) = \{g \in G \mid g\alpha = \alpha g\}$$

for the centralizer of α in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid gHg^{-1} = H\}$$

for the normalizer of H in G.

Notation 1.2. Let $S^{\log}$ be an fs log scheme (cf. [Naka], Definition 1.7).

(i) Write S for the underlying scheme of $S^{\log}$.

(ii) Write $\mathcal{M}_S$ for the sheaf of monoids that defines the log structure of $S^{\log}$.

(iii) Let $\overline{s}$ be a geometric point of S. Then we shall denote by $I(\overline{s}, \mathcal{M}_S)$ the ideal of $\mathcal{O}_{S,\overline{s}}$ generated by the image of $\mathcal{M}_{S,\overline{s}} \setminus \mathcal{O}^{\times}_{S,\overline{s}}$ via the homomorphism of monoids $\mathcal{M}_{S,\overline{s}} \to \mathcal{O}_{S,\overline{s}}$ induced by the morphism $\mathcal{M}_S \to \mathcal{O}_S$ which defines the log structure of $S^{\log}$.

(iv) Let $s \in S$ and $\overline{s}$ a geometric point of S which lies over s. Write $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\text{gp}}$ for the groupification of $\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}}$. Then we shall refer to the rank of the finitely generated free abelian group $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\text{gp}}$ as the *log rank* at s. Note that one verifies easily that this rank is independent of the choice of $\overline{s}$, i.e., depends only on s.

(v) Let $m \in \mathbb{Z}$. Then we shall write

$$S^{\log \leq m} \stackrel{\text{def}}{=} \{s \in S \mid \text{the log rank at } s \text{ is} \leq m\}.$$

Note that since $S^{\log\leq m}$ is open in S (cf. [MzTa], Proposition 5.2, (i)), we shall also regard (by abuse of notation) $S^{\log\leq m}$ as an open subscheme of S.

(vi) We shall write $U_S \stackrel{\text{def}}{=} S^{\log\leq 0}$ and refer to U_S as the *interior* of $S^{\log}$. When $U_S = S$, we shall often use the notation S to denote the log scheme $S^{\log}$.

Notation 1.3. Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$.

(i) Write $\overline{\mathcal{M}}_{g,r}$ for the moduli stack (over $\mathbb{Z}$) of pointed stable curves of type (g, r) and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ for the open substack corresponding to the smooth curves. Here, we assume the marked points to be ordered.

(ii) Write

$$\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

for the tautological curve over $\overline{\mathcal{M}}_{g,r}$; $\overline{\mathcal{D}}_{g,r} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ for the divisor at infinity.

(iii) Write $\overline{\mathcal{M}}^{\log}_{g,r}$ for the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ with the log structure determined by the divisors with normal crossings $\overline{\mathcal{D}}_{g,r}$.

(iv) The divisor of $\overline{\mathcal{C}}_{g,r}$ given by the union of $\overline{\mathcal{C}}_{g,r} \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{D}}_{g,r}$ with the divisor of $\overline{\mathcal{C}}_{g,r}$ determined by the marked points determines a log structure on $\overline{\mathcal{C}}_{g,r}$; we denote the resulting log stack by $\overline{\mathcal{C}}_{g,r}^{\log}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}_{g,r}^{\log}$. If $S^{\log}$ is an arbitrary log scheme, then we shall refer to a morphism

$$C^{\log} \to S^{\log}$$

whose pull-back to some finite étale covering $T \to S$ is isomorphic to the pull-back of the tautological log curve via some morphism $T^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S T \to \overline{\mathcal{M}}_{g,r}^{\log}$ as a *stable log curve* (of type (g,r)). If $C \to S$ is smooth, i.e., every geometric fiber of $C \to S$ is free of nodes, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve* (of type (g,r)).

(v) A smooth log curve of type $(0,3)$ will be referred to as a *tripod*. A vertex of a semi-graph of anabelioids of pro-l PSC-type (cf. [CmbGC], Definition 1.1, (i)) of type $(0,3)$ (cf. [CbTpI], Definition 2.3, (iii)) will also be referred to as a *tripod*.

Definition 1.4. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type (cf. [CmbGC], Definition 1.1, (i)) and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. Write

$$\mathrm{Cusp}(\mathbb{G}) \ (\text{resp. } \mathrm{Node}(\mathbb{G}),\ \mathrm{Vert}(\mathbb{G}),\ \mathrm{Edge}(\mathcal{G}))$$

for the set of cusps (resp. nodes, vertices, edges) of $\mathbb{G}$ and

$$\mathrm{Cusp}(\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Cusp}(\mathbb{G}),\ \mathrm{Node}(\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Node}(\mathbb{G}),$$

$$\mathrm{Vert}(\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Vert}(\mathbb{G}),\ \mathrm{Edge}(\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Edge}(\mathbb{G}).$$

2. **Log configuration spaces and log divisors**

Let l be a prime number; k an algebraically closed field of characteristic $\neq l$; $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$;

$$X^{\log} \to S$$

(cf. Notation 1.2, (vi)) a smooth log curve of type (g,r); $n \in \mathbb{Z}_{>0}$. We suppose that the marked points of $X^{\log}$ are equipped with an ordering, and that

$$r > 0$$

(cf. the discussion at the end of the Introduction). In the present §2, we define log configuration spaces, log-full points, and log divisors.

Definition 2.1. The smooth log curve $X^{\log}$ over S determines, up to a choice of ordering of the marked points (which will in fact *not* affect the following construction), a *classifying morphism* $S \to \overline{\mathcal{M}}_{g,r}^{\log}$. Thus, by pulling back the morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ given by forgetting the last n marked points via this morphism $S \to \overline{\mathcal{M}}_{g,r}^{\log}$, we obtain a morphism of log schemes

$$X_n^{\log} \to S.$$

We shall refer to $X_n^{\log}$ as the *n-th log configuration space associated to* $X^{\log} \to S$. Note that $X_1^{\log} = X^{\log}$. Write $X_0^{\log} \stackrel{\mathrm{def}}{=} S$.

Definition 2.2. (i) Write

$$\Pi_n \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l}(X_n^{\log})$$

for the maximal pro-l quotient of the fundamental group of the log scheme $X_n^{\log}$ (for a suitable choice of basepoint). We refer to [Hsh], Theorems B.1, B.2, for more details on fundamental groups of log schemes.

(ii) Let P be a closed point of X_n. By abuse of notation, we shall use the notation"P" *both* for the corresponding point of the scheme X_n *and* for the reduced closed subscheme of X_n determined by this point. Then we shall say that P is a *log-full point* of $X_n^{\log}$ if

$$\dim(\mathcal{O}_{X_n,P}/I(P, \mathcal{M}_{X_n})) = 0$$

(cf. Notation 1.2, (iii)).

(iii) Let P be a log-full point of $X_n^{\log}$ and $P^{\log}$ the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the reduced closed subscheme of X_n determined by P. Then we obtain an outer homomorphism $\pi_1(P^{\log}) \to \Pi_n$ (for suitable choices of basepoints). We shall refer to the subgroup $\mathrm{Im}(\pi_1(P^{\log}) \to \Pi_n)$, which is well-defined up to Π_n-conjugation, as a *log-full subgroup* at P.

(iv) We shall often refer to a point of the scheme X_n as a *point of* $X_n^{\log}$. Let P be a point of $X_n^{\log}$. Then P parametrizes a pointed stable curve of type $(g, r+n)$ (cf. Definition 2.1). Thus, any geometric point of $X_n^{\log}$ lying over P determines a semi-graph of anabelioids of pro-l PSC-type, which is in fact easily verified to be independent of the choice of geometric point lying over P. We shall write $\mathcal{G}_P$ for this semi-graph of anabelioids of pro-l PSC-type.

(v) Let us fix an ordered set

$$C_{r,n} \stackrel{\mathrm{def}}{=} \{c_1, \ldots, c_r, x_1 \stackrel{\mathrm{def}}{=} c_{r+1}, \ldots x_n \stackrel{\mathrm{def}}{=} c_{r+n}\}.$$

Thus, by definition, for each point P of $X_n^{\log}$, we have a natural bijection $C_{r,n} \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_P)$. In the following, let us identify the set $\mathrm{Cusp}(\mathcal{G}_P)$ with $C_{r,n}$.

(vi) We shall refer to an irreducible divisor of X_n contained in the complement $X_n \setminus U_{X_n}$ of the interior U_{X_n} of $X_n^{\log}$ as a *log divisor* of $X_n^{\log}$. That is to say, a log divisor of $X_n^{\log}$ is an irreducible divisor of X_n whose generic point parametrizes a pointed stable curve with precisely two irreducible components (cf. Definition 2.1).

(vii) Let V be a log divisor of $X_n^{\log}$. Then we shall write $\mathcal{G}_V$ for "$\mathcal{G}_P$" in the case where we take "P" to be the generic point of V.

(viii) For each $i \in \{1, \ldots, n\}$, write $p_i \colon X_n^{\log} \to X^{\log}$ for the projection morphism of co-profile $\{i\}$ (cf. [MzTa], Definition 2.1, (ii)). Write $\iota \stackrel{\mathrm{def}}{=} (p_1, \ldots, p_n) \colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$.

Definition 2.3. Let $m \geq 2$ and $y_1, \ldots, y_m \in C_{r,n}$ distinct elements such that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$. Then one verifies immediately — by considering *clutching morphisms* (cf. [Knu], Definition 3.8) — that there exists a unique log divisor V of $X_n^{\log}$, which we shall denote by $V(\{y_1, \ldots, y_m\})$, that satisfies the following condition: $\mathcal{G}_V$ has precisely two vertices v_1, v_2 such that v_1 is of type

$(0, m+1)$, v_2 is of type $(g, n+r-m+1)$, and $y_1, \ldots, y_m$ are cusps of $\mathcal{G}_V|_{v_1}$ (cf. [CbTpI], Definition 2.1, (iii)).

Remark 2.4. Let V be a log divisor of $X_n^{\log}$. Then let us *observe* that there exists a unique collection of distinct elements $y_1, \ldots, y_m \in C_{r,n}$ such that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$ and $V = V(\{y_1, \ldots, y_m\})$. (Note that uniqueness holds even in the case where $g = 0$ (in which case $r \geq 3$), as a consequence of the condition that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$.) This observation is essentially a special case of Proposition 2.6, (iii), below.

Definition 2.5. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. Suppose that $\mathbb{G}$ is a tree. Let $e \in \mathrm{Edge}(\mathcal{G})$, $v \in \mathrm{Vert}(\mathcal{G})$ be such that e abuts to v. Write b for the branch of e that abuts to v. By replacing e by open edges e_1, e_2 such that e_1 abuts to v, and e_2 abuts to the vertex $\neq v$ to which e abuts (resp. e_1 abuts to v, and e_2 is an edge which abuts to no vertex) if $e \in \mathrm{Node}(\mathcal{G})$ (resp. $e \in \mathrm{Cusp}(\mathcal{G})$), we obtain two connected semi-graphs. Write $\mathbb{G}_{\not\ni b}$ for the semi-graph (among these two connected semi-graphs) that does not contain b. Write $\mathbb{G}_{\ni b}$ for the semi-graph (among these two connected semi-graphs) that contains b. Observe that

- for arbitrary $e \in \mathrm{Edge}(\mathcal{G})$, $\mathcal{G}$ determines a natural semi-graph of anabelioids of pro-l PSC-type $\mathcal{G}_{\ni b}$ whose underlying semi-graph may be identified with $\mathbb{G}_{\ni b}$;
- if $e \in \mathrm{Node}(\mathcal{G})$, then $\mathcal{G}$ also determines a natural semi-graph of anabelioids of pro-l PSC-type $\mathcal{G}_{\not\ni b}$ whose underlying semi-graph may be identified with $\mathbb{G}_{\not\ni b}$.

Proposition 2.6. *Let P be a point of $X_n^{\log}$. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}_P$ (cf. Definition 2.2, (iv)). Then the following hold:*

(i) $\mathbb{G}$ *is a tree.*
(ii) $\mathrm{Cusp}(\mathcal{G}_P) = \{c_1, \ldots, c_r, x_1, \ldots, x_n\}$.
(iii) There exists a unique vertex $v_g \in \mathrm{Vert}(\mathcal{G}_P)$ that satisfies the following properties:
  (a) The genus of $\mathcal{G}_P|_{v_g}$ (cf. [CbTpI]*, Definitions 2.1, (iii); 2.3, (ii)) is g.*
  (b) Let $e \in \mathrm{Node}(\mathcal{G}_P)$ that abuts to v_g and b_g the branch of e that abuts to v_g. Then $\sharp(\mathrm{Cusp}((\mathbb{G})_{\ni b_g}) \cap \{c_1, \ldots, c_r\}) \geq r-1$.
  (c) For each $v \in \mathrm{Vert}(\mathcal{G}_P) \setminus \{v_g\}$, the genus of $\mathcal{G}_P|_v$ is 0.

Proof. Assertion (i) follows immediately from the definition of $\mathcal{G}_P$. Assertion (ii) follows from Definition 2.2, (v). Finally, we verify assertion (iii). Existence is immediate. If $g \neq 0$, uniqueness is immediate. If $g = 0$, it follows that $r \geq 3$. Now assume that there exists a vertex $v_g' \in \mathrm{Vert}(\mathcal{G}_P)$ such that $v_g' \neq v_g$, and v_g' satisfies conditions (a), (b). It follows immediately from the connectedness of $\mathbb{G}$ that there exists a node $e \in \mathrm{Node}(\mathcal{G}_P)$ such that e abuts to v_g, and $v_g' \in \mathrm{Vert}(\mathbb{G}_{\not\ni b_g})$, where we write b_g for the branch of e that abuts to v_g. By condition (b) in the case of v_g, b_g, it holds that $\sharp(\mathrm{Cusp}(\mathbb{G}_{\ni b_g}) \cap \{c_1, \ldots, c_r\}) \geq r-1 \geq 2$. On the other hand, it follows immediately from the connectedness of $\mathbb{G}$ that there exists a node $e' \in \mathrm{Node}(\mathcal{G}_P)$ such that e' abuts to v_g', and $v_g \in \mathrm{Vert}(\mathbb{G}_{\not\ni b_g'})$, where we write b_g' for the branch of e' that abuts to v_g'. Next observe that it follows immediately from the fact that $\mathbb{G}$ is a tree that $\mathbb{G}_{\ni b_g}$ is a sub-semi-graph of $\mathbb{G}_{\not\ni b_g'}$, which implies that

$\sharp(\mathrm{Cusp}(\mathbb{G}_{\ni b'_g}) \cap \{c_1, \ldots, c_r\}) \leq r-2$. Thus, by condition (b) in the case of v'_g, b'_g, we obtain a contradiction. $\square$

Definition 2.7. Let P be a point of $X_n^{\log}$. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}_P$, $v_g \in \mathrm{Vert}(\mathcal{G}_P)$ for the vertex of Proposition 2.6, (iii). For $e \in \mathrm{Node}(\mathcal{G}_P)$, write b_e for the branch of e such that $v_g \in \mathrm{Vert}(\mathbb{G}_{\ni b_e})$. Then we shall write

$$I_{\mathbb{G}} \stackrel{\mathrm{def}}{=} \{\mathrm{Cusp}((\mathbb{G})_{\not\ni b_e}) \cap C_{r,n} \mid e \in \mathrm{Node}(\mathcal{G}_P)\} \subseteq 2^{C_{r,n}},$$

where we write $2^{(-)}$ for the set of subsets of $(-)$. Note that it follows immediately from Proposition 2.6, (iii), that for each $I \in I_{\mathbb{G}}$, $\sharp I \geq 2$.

Proposition 2.8. *Let P, P' be points of $X_n^{\log}$. Write $\mathbb{G}$, $\mathbb{G}'$ for the respective underlying semi-graphs of $\mathcal{G}_P$, $\mathcal{G}_{P'}$; v_g, v'_g for the respective vertices characterized in Proposition 2.6, (iii). If $I_{\mathbb{G}} = I_{\mathbb{G}'} \subseteq 2^{C_{r,n}}$, then there exists a unique isomorphism of semi-graphs $\mathbb{G} \xrightarrow{\sim} \mathbb{G}'$ that maps $v_g \mapsto v'_g$ and is compatible with the labels of cusps $\in C_{r,n}$. Moreover, $\sharp\mathrm{Vert}(\mathbb{G}) = \sharp I_{\mathbb{G}} + 1$, $\sharp\mathrm{Node}(\mathbb{G}) = \sharp\mathrm{Node}(\mathcal{G}_P) = \sharp I_{\mathbb{G}}$.*

Proof. Let $J \in I_{\mathbb{G}}$. Write $J_{\subseteq} \stackrel{\mathrm{def}}{=} \{I \in I_{\mathbb{G}} \mid I \subseteq J \subseteq C_{r,n}\}$. Then one verifies immediately that one may construct a (well-defined) *semi-graph* $\mathbb{G}_J$ satisfying the following properties:

(i) The elements of $\mathrm{Vert}(\mathbb{G}_J)$ are equipped with labels $\in J_{\subseteq}$ that determine a bijection $\mathrm{Vert}(\mathbb{G}_J) \xrightarrow{\sim} J_{\subseteq}$.

(ii) Let us call a subset $\{J_1, J_2\} \subseteq J_{\subseteq}$ of cardinality ≤ 2 an *adjacent pair* of $J_{\subseteq}$ if $J_1 \subsetneq J_2$, and there does not exist an element $I \in I_{\mathbb{G}}$ such that $J_1 \subsetneq I \subsetneq J_2$. For $e \in \mathrm{Node}(\mathbb{G}_J)$, write $\mathrm{Vert}(e) \subseteq \mathrm{Vert}(\mathbb{G}_J) \xrightarrow{\sim} J_{\subseteq}$ for the subset (of cardinality ≤ 2) of vertices to which e abuts. Then the assignment

$$\mathrm{Node}(\mathbb{G}_J) \ni e \mapsto \mathrm{Vert}(e) \in 2^{\mathrm{Vert}(\mathbb{G}_J)} \xrightarrow{\sim} 2^{J_{\subseteq}}$$

determines a bijection of $\mathrm{Node}(\mathbb{G}_J)$ onto the set of adjacent pairs of $J_{\subseteq}$.

(iii) The cusps of $\mathbb{G}_J$ are equipped with labels $\in C_{r,n}$ in such a way that, for each $I \in J_{\subseteq}$, these labels determine a bijection from the set of cusps of the vertex labeled by I onto the subset $I \setminus (\bigcup_{I_{\mathbb{G}} \ni J^* \subsetneq I} J^*) \subseteq C_{r,n}$. Moreover, these labels determine a bijection $\mathrm{Cusp}(\mathbb{G}_J) \xrightarrow{\sim} J$ $(\subseteq C_{r,n})$.

Next, one verifies immediately that one may construct a (well-defined) *semi-graph* $\mathbb{G}_{I_{\mathbb{G}}}$ satisfying the following properties:

(I) There exists a unique vertex of $\mathbb{G}_{I_{\mathbb{G}}}$ equipped with a label v_g. The set of cusps of this vertex v_g are equipped with labels $\in C_{r,n}$ which determine a bijection from the set of cusps of this vertex v_g to the subset $C_{r.n} \setminus (\bigcup_{I \in I_{\mathbb{G}}} I) \subseteq C_{r,n}$.

(II) The semi-graph $\mathbb{G}_{I_{\mathbb{G}}}$ is obtained from v_g (together with its associated cusps) by *gluing* v_g to $\mathbb{G}_J$, where $J \in I_{\mathbb{G}}$ ranges over the elements of $I_{\mathbb{G}}$ that are *maximal* with respect to the relation of inclusion, along a node $e_J \in \mathrm{Node}(\mathbb{G}_{I_{\mathbb{G}}})$ that abuts to v_g and the vertex of $\mathbb{G}_J$ labeled J (cf. (i)).

(III) The cusps of $\mathbb{G}_{I_{\mathbb{G}}}$ are equipped with labels $\in C_{r,n}$ that are compatible with the labels of (I) (in the case of the cusps associated to the vertex labeled v_g) and (i) (in the case of the cusps associated to vertices $\in \mathrm{Vert}(\mathbb{G}_J)$, for J as in (II)). These labels determine a bijection $\mathrm{Cusp}(\mathbb{G}_{I_{\mathbb{G}}}) \xrightarrow{\sim} C_{r,n}$.

Then it follows immediately from Definition 2.7 that there exists a unique isomorphism of semi-graphs $\mathbb{G} \xrightarrow{\sim} \mathbb{G}_{I_{\mathbb{G}}}$ that is compatible with the label "v_g", as well as with the labels of cusps $\in C_{r,n}$. Since $\mathbb{G}$ is a tree, it follows that $\mathbb{G}_{I_{\mathbb{G}}} \stackrel{\sim}{\leftarrow} \mathbb{G}$ is also a tree. On the other hand, observe that it follows immediately from the construction of $\mathbb{G}_{I_{\mathbb{G}}}$ (cf. (i), (I), (II)), together with the definition of $I_{\mathbb{G}}$ (cf. Definition 2.7), that $\sharp\mathrm{Vert}(\mathbb{G}_{I_{\mathbb{G}}}) = \sharp I_{\mathbb{G}} + 1$. Since $\mathbb{G}_{I_{\mathbb{G}}}$ is a tree, we thus conclude that $\sharp\mathrm{Node}(\mathbb{G}_{I_{\mathbb{G}}}) = \sharp I_{\mathbb{G}}$. Finally, since $\mathbb{G}_{I_{\mathbb{G}}}$ is *completely determined* by the subset $I_{\mathbb{G}} \subseteq 2^{C_{r,n}}$, the remainder of Proposition 2.8 follows immediately. $\square$

Proposition 2.9. *Let P be a point of $X_n^{\log}$ and $I \subseteq C_{r,n}$ such that $\sharp(I \cap \{c_1, \ldots, c_r\}) \leq 1$. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}_P$. Then the following conditions are equivalent:*

(i) $P \in V(I)$ (cf. Definition 2.3).
(ii) $I \in I_{\mathbb{G}}$.
(iii) $\mathcal{G}_{V(I)}$ is obtained from $\mathcal{G}_P$ by generization (with respect to some subset of $\mathrm{Node}(\mathcal{G}_P)$ *(cf.* [CbTpI], *Definition 2.8)).*

Proof. The equivalence (i) $\Longleftrightarrow$ (iii) follows immediately — by considering *clutching morphisms* (cf. [Knu], Definition 3.8) — from the latter portion of Definition 2.2, (vi). The equivalence (ii) $\Longleftrightarrow$ (iii) follows immediately from Definition 2.7. $\square$

Proposition 2.10. *Let $m \in \{1, \ldots, n\}$; $V_1, \ldots, V_m$ a collection of distinct log divisors of $X_n^{\log}$ such that $V_1 \cap \cdots \cap V_m \neq \emptyset$. Then there exist nonnegative integers $i_0, \ldots, i_m$ such that*

$$i_0 + \cdots + i_m = n - m,$$

and the intersection $V_1 \cap \cdots \cap V_m$ is isomorphic, over S, to

$$X_{i_0} \times_S (\overline{\mathcal{M}}_{0,i_1+3} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \overline{\mathcal{M}}_{0,i_m+3} \times_{\mathbb{Z}} S).$$

In particular, the intersection $V_1 \cap \cdots \cap V_m$ is irreducible of dimension $n - m$; if $m = n$, then $V_1 \cap \cdots \cap V_n$ is (the reduced closed subscheme determined by) a log-full point.

Proof. Let P be a generic point of $V_1 \cap \cdots \cap V_m$. Write $\mathbb{G}_P$ for the underlying semi-graph of $\mathcal{G}_P$. Recall from Proposition 2.8 that $\sharp\mathrm{Vert}(\mathbb{G}_P) - 1 = \sharp\mathrm{Node}(\mathcal{G}_P) = \sharp I_{\mathbb{G}_P}$. Thus, we conclude from Remark 2.4, together with the equivalence (i) $\Longleftrightarrow$ (ii) of Proposition 2.9, that

$$\sharp\mathrm{Node}(\mathcal{G}_P) = \sharp I_{\mathbb{G}_P} = \sharp\{V \mid V \text{ is a log divisor of } X_n^{\log} \text{ such that } P \in V\} \geq m.$$

Since the divisor that determines the log structure of $X_n^{\log}$ is a divisor with normal crossings, we thus conclude that $\sharp\mathrm{Vert}(\mathbb{G}_P) - 1 = \sharp\mathrm{Node}(\mathcal{G}_P) = m$, and hence that

$$\{V \mid V \text{ is a log divisor of } X_n^{\log} \text{ such that } P \in V\} = \{V_1, \ldots, V_m\}.$$

Thus, it follows from Proposition 2.6, (ii), that

$$\sharp\{\text{branches of edges (i.e., cusps and nodes) of } \mathcal{G}_P\} = n + r + 2m.$$

Next, observe that it follows from Proposition 2.6, (iii), that there exists a *clutching morphism*

$$\rho_P \colon X_{i_0} \times_S (\overline{\mathcal{M}}_{0,i_1+3} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \overline{\mathcal{M}}_{0,i_m+3} \times_{\mathbb{Z}} S) \to X_n$$

(cf. [Knu], Definition 3.8) such that P lies in the image of this morphism ρ_P. Since the morphism ρ_P is a proper monomorphism (cf. Propositions 2.6, (iii); 2.8), it follows that the morphism ρ_P is a closed immersion. Thus, if we write X_P for the scheme-theoretic closure of P in X_n and X_{ρ_P} for the image of ρ_P in X_n, then $X_P \subseteq X_{\rho_P}$.

Next, observe that since the sum of the cardinalities of the sets of cusps of the pointed stable curves parametrized by the moduli stack factors of the domain of ρ_P is equal to

$$(i_0 + r) + \sum_{j=1}^{m} (i_j + 3),$$

it holds that

$$(i_0 + r) + \sum_{j=1}^{m} (i_j + 3) = \sharp\{\text{branches of edges of } \mathcal{G}_P\} = n + r + 2m,$$

and hence that

$$i_0 + \sum_{j=1}^{m} i_j = n - m.$$

Since

$$\dim(X_{\rho_P}) = \dim(X_{i_0} \times_S (\overline{\mathcal{M}}_{0,i_1+3} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} \overline{\mathcal{M}}_{0,i_m+3} \times_{\mathbb{Z}} S)) = i_0 + \sum_{j=1}^{m} i_j,$$

and $V_1 \cup \cdots \cup V_m$ is a divisor with normal crossings (which implies that $\dim(X_P) = \dim(V_1 \cap \cdots \cap V_m) = n - m$), we thus conclude that

$$\dim(X_{\rho_P}) = \dim(X_P),$$

and hence that $X_{\rho_P} = X_P$. Moreover, since

$$\{V \mid V \text{ is a log divisor of } X_n^{\log} \text{ such that } P \in V\} = \{V_1, \ldots, V_m\},$$

we thus conclude from Remark 2.4, together with the equivalence (i) $\Longleftrightarrow$ (ii) of Proposition 2.9, that $\{V_1, \ldots, V_m\}$ determines $I_{\mathbb{G}_P}$, hence, by Proposition 2.8, that $\{V_1, \ldots, V_m\}$ determines $\mathcal{G}_P$. But this implies that every generic point of $V_1 \cap \cdots \cap V_m$ lies in X_{ρ_P}, for some fixed P, and hence that $V_1 \cap \cdots \cap V_m$ is irreducible. This completes the proof of Proposition 2.10. $\square$

3. **Various types of log divisors**

We continue with the notation introduced at the beginning of §2. In addition, we suppose that $n \in \mathbb{Z}_{>1}$. In the present §3, we define various types of log divisors and study their geometry.

Definition 3.1. (i) For positive integers $i \in \{1, \ldots, n-1\}$, $j \in \{i+1, \ldots, n\}$, write

$$\pi_{i,j} \colon X \times_S \cdots \times_S X \to X \times_S X$$

for the projection of the fiber product of n copies of $X \to S$ to the i-th and j-th factors. Write $\delta'_{i,j}$ for the inverse image via $\pi_{i,j}$ of the image of the diagonal embedding $X \hookrightarrow X \times_S X$. Write $\delta_{i,j}$ for the uniquely determined log divisor of $X_n^{\log}$ whose generic point maps to the generic point of $\delta'_{i,j}$ via the natural morphism $X_n \to X \times_S \cdots \times_S X$ (cf. Definition 2.2, (viii)). We shall refer to the log divisor $\delta_{i,j}$ as a *naive diagonal* of $X_n^{\log}$.

(ii) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *tripodal divisor* if one of the vertices of $\mathcal{G}_V$ (cf. Definition 2.2, (vii)) is a tripod (cf. Notation 1.3, (v)).

(iii) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *(g,r)-divisor* if one of the vertices of $\mathcal{G}_V$ is of type (g,r) (cf. [CbTpI], Definition 2.3, (iii)).
(iv) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *drift diagonal* if there exist a naive diagonal δ and an automorphism α of $X_n^{\log}$ over S such that $V = \alpha(\delta)$.

Remark 3.2. Recall (cf. [NaTa], Theorem D) that:

- when $(g,r) = (0,3)$ or $(1,1)$, any automorphism of $X_n^{\log}$ over S necessarily arises as the composite of an automorphism (of $X_n^{\log}$ that arises from an automorphism) of $X^{\log}$ over S with an automorphism of $X_n^{\log}$ that arises from a permutation of the $r+n$ marked points of the stable log curve $X_{n+1}^{\log} \to X_n^{\log}$;
- when $(g,r) \neq (0,3), (1,1)$, any automorphism of $X_n^{\log}$ over S necessarily arises as the composite of an automorphism (of $X_n^{\log}$ that arises from an automorphism) of $X^{\log}$ over S with an automorphism of $X_n^{\log}$ that arises from a permutation of the n factors of $X_n^{\log}$.

Proposition 3.3. *The following hold:*

(i) It holds that

$$\{\textit{naive diagonals}\} = \{V(\{x_i, x_j\}) \mid i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}\}$$

(cf. Definition 2.3).

(ii) If $(g,r) \neq (0,3)$, then

$$\{\textit{tripodal divisors}\}$$
$$= \{V(\{y_1, y_2\}) \mid y_1, y_2 \in C_{r,n} \textit{ are distinct elements}, \{y_1, y_2\} \not\subseteq \{c_1, \ldots, c_r\}\}$$

(cf. Definition 2.3).

(iii) If $(g,r) = (0,3)$, then

$$\{\textit{tripodal divisors}\} = \{V(\{y_1, y_2\}) \mid C_{r,n} \supseteq \{y_1, y_2\} \not\subseteq \{c_1, c_2, c_3\}\}$$
$$\cup \ \{V(\{y_1, y_2\}) \stackrel{\mathrm{def}}{=} V(C_{r,n} \setminus \{y_1, y_2\}) \mid \{y_1, y_2\} \subseteq \{c_1, c_2, c_3\}\}$$

(cf. Definition 2.3).

(iv) Let V be a tripodal divisor and α an automorphism of $X_n^{\log}$ over S. Then $\alpha(V)$ is a tripodal divisor.

Proof. First, assertion (i) follows immediately from the various definitions involved. Next, assertions (ii), (iii) follow immediately from Remark 2.4, together with the definition of tripodal divisors. Finally, we consider assertion (iv). It follows from Remark 3.2 that α lifts to an automorphism of $X_{n+1}^{\log}$ relative to the natural morphism $X_{n+1}^{\log} \to X_n^{\log}$, hence induces an isomorphism of $\mathcal{G}_V$ with $\mathcal{G}_{\alpha(V)}$. This completes the proof of assertion (iv). □

Proposition 3.4. *The following hold:*

(i) It holds that

$$\{\textit{naive diagonals}\} \subseteq \{\textit{drift diagonals}\} \subseteq \{\textit{tripodal divisors}\} \subseteq \{\textit{log divisors}\}.$$

(ii) If $(g,r) \neq (0,3), (1,1)$, then

$$\{\textit{naive diagonals}\} = \{\textit{drift diagonals}\}.$$

(iii) If $(g, r) = (0, 3)$ *or* $(1, 1)$*, then*

$$\{\textit{drift diagonals}\} = \{\textit{tripodal divisors}\}.$$

Proof. First, we verify assertion (i). The first and third inclusions follow immediately from the various definitions involved. The second inclusion follows from Proposition 3.3, (i), (iv). This completes the proof of assertion (i). Assertion (ii) follows immediately from Remark 3.2.

Finally, we consider assertion (iii). Let V be a tripodal divisor. Let us first suppose that $(g, r) = (0, 3)$. Then $X_n^{\log}$ is naturally isomorphic to the moduli stack $(\overline{\mathcal{M}}_{0,n+3}^{\log})_k \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{0,n+3}^{\log} \times_{\mathbb{Z}} S$ over S, on which the symmetric group on $n + 3$ letters acts naturally. Moreover, it follows from Proposition 3.3, (iii), that $V = V(\{y_1, y_2\})$, where $y_1, y_2 \in C_{r,n}$ are distinct elements. Thus, there exists a permutation $\alpha \in S_{n+3}$ such that $\alpha(V(\{x_1, x_2\})) = V(\{y_1, y_2\})$. Assertion (iii) in the case where $(g, r) = (0, 3)$ now follows immediately.

Next, let us suppose that $(g, r) = (1, 1)$. Then $X_n^{\log}$ is naturally isomorphic to the fiber product $\overline{\mathcal{M}}_{1,n+1}^{\log} \times_{\overline{\mathcal{M}}_{1,1}^{\log}} S$ over S, where the arrow $S \to \overline{\mathcal{M}}_{1,1}^{\log}$ is taken to be the classifying morphism $S \to \overline{\mathcal{M}}_{1,1}^{\log}$ determined by $X^{\log}$ (cf. Definition 2.1). Thus, one verifies easily, by considering the automorphisms of an elliptic curve given by translation by a rational point, that the action of the symmetric group on $n + 1$ letters on $\overline{\mathcal{M}}_{1,n+1}^{\log}$ induces an action of the symmetric group on $n + 1$ letters on $X_n^{\log}$. Moreover, it follows from Proposition 3.3, (ii), that $V = V(\{y_1, y_2\})$, where $y_1, y_2 \in C_{r,n}$ are *arbitrary* distinct elements (cf. Definition 2.3). Thus, there exists a permutation $\alpha \in S_{n+1}$ such that $\alpha(V(\{x_1, x_2\})) = V(\{y_1, y_2\})$. Assertion (iii) in the case where $(g, r) = (1, 1)$ now follows immediately. □

Definition 3.5. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type.

(i) We shall say that a vertex of $\mathcal{G}$ is a *terminal vertex* if precisely one node abuts to it.
(ii) We shall say that a node of $\mathcal{G}$ is a *terminal node* if it abuts to a terminal vertex.
(iii) Write

$$\mathrm{TerNode}(\mathcal{G}) \subseteq \mathrm{Node}(\mathcal{G})$$

for the set of terminal nodes of $\mathcal{G}$.

Proposition 3.6. *Let P be a closed point of $X_n^{\log}$. Then it holds that*

$$P \textit{ is a log-full point} \Longleftrightarrow \mathrm{Node}(\mathcal{G}_p) = n.$$

Proof. This equivalence follows immediately from Definitions 2.1, 2.2, (ii), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ (where we recall that the log structure of this log stack arises from a divisor with normal crossings). □

Proposition 3.7. *Let P be a log-full point of $X_n^{\log}$ and A a log-full subgroup at P (cf. Definition 2.2, (iii)). Then the following hold:*

(i) It holds that $\sharp\mathrm{Node}(\mathcal{G}_P) = n$. The underlying semi-graph of $\mathcal{G}_P$ is a tree that has precisely $n + 1$ vertices, one of which is of type (g, r) (cf. [CbTpI], *Definition 2.3, (iii)); the other vertices are tripods (cf. Notation 1.3, (v)).*

(ii) *Write* $\mathrm{Node}(\mathcal{G}_P) = \{e_1, \ldots, e_n\}$ *(cf. (i)). Then for each* $i \in \{1, \ldots, n\}$*, there exists a unique log divisor* V_i *such that there exists an isomorphism of* $\mathcal{G}_{V_i}$ *with* $(\mathcal{G}_P)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_P)\setminus\{e_i\}}$ *(cf.* [CbTpI], *Definition 2.8) which preserves the respective orderings of cusps. In this situation, we shall say that* V_i *is the log divisor associated to* $e_i \in \mathrm{Node}(\mathcal{G}_P)$*.*

(iii) *In the situation of (ii),*

$$P = V_1 \cap \cdots \cap V_n \textit{ and } A = I_{V_1} \times \cdots \times I_{V_n},$$

where $I_{V_i} \subseteq \Pi_n$ *is a suitable inertia group associated to* V_i *contained in* A*. Moreover, for each* $i \in \{1, \ldots, n\}$*, it holds that* $I_{V_i} \simeq \mathbb{Z}_l$ *and* $A \simeq \mathbb{Z}_l^{\oplus n}$*.*

(iv) *Let* m *be a positive number;* $W_1, \ldots, W_m$ *distinct log divisors such that* $P = W_1 \cap \cdots \cap W_m$*. Then* $m = n$*, and* $\{W_1, \ldots, W_m\} = \{V_1, \ldots, V_n\}$ *(cf. (iii)).*

Proof. Assertion (i) follows immediately from Propositions 2.6, 2.8, and 3.6, together with the *observation* that a *log-full point* (cf. Definition 2.2, (ii)) corresponds to an *intersection* of the sort considered in Proposition 2.10, in the case where $n = m$, and $i_j = 0$, for $j = 0, 1, \ldots, m$. Assertion (ii) follows immediately from Proposition 2.9. Assertion (iii) follows from Propositions 2.9 and 2.10, and [CbTpI], Lemma 5.4, (ii). Assertion (iv) follows immediately from Propositions 2.8, 2.9, 2.10, together with assertion (iii). □

Definition 3.8. Let P be a log-full point of $X_n^{\log}$ and $V_1, \ldots, V_n$ the log divisors such that $P = V_1 \cap \cdots \cap V_n$ (cf. Proposition 3.7, (iv)). We shall say that V_i is a *terminal divisor* of P if there exists a terminal node $e \in \mathrm{TerNode}(\mathcal{G}_P)$ such that V_i is the log divisor associated to $e \in \mathrm{Node}(\mathcal{G}_P)$ (cf. Proposition 3.7, (ii)).

Lemma 3.9. *Let* P *be a log-full point of* $X_n^{\log}$ *and* $V_1, \ldots, V_n$ *the log divisors such that* $P = V_1 \cap \cdots \cap V_n$ *(cf. Proposition 3.7, (iv)). Then the following conditions are equivalent:*

(i) V_i *is a terminal divisor of* P*.*

(ii) V_i *is a tripodal divisor or a* (g, r)*-divisor.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 3.7, (i), (ii). The implication (ii) $\Longrightarrow$ (i) follows immediately from the various definitions involved. □

Theorem 3.10. *For* $\square \in \{\circ, \bullet\}$*, let* $l^\square$ *be a prime number;* $k^\square$ *an algebraically closed field of characteristic* $\neq l^\square$*;* $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$*;* $(g^\square, r^\square)$ *a pair of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$*;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$*;* $n^\square \in \mathbb{Z}_{>1}$*;* $X_{n^\square}^{\log \square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log \square} \to S^\square$*;* $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log \square})$ *(for a suitable choice of basepoint);*

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. Then the following hold:

(i) $(g^\circ, r^\circ, n^\circ) = (g^\bullet, r^\bullet, n^\bullet)$*.*

(ii) *If* $(g^\square, r^\square) \neq (0, 3), (1, 1)$*, then* ϕ *induces a bijection between the set of fiber subgroups of a given co-length (cf.* [MzTa], *Definition 2.3, (iii)) of* Π° *and the set of fiber subgroups of the same co-length of* $\Pi^\bullet$*.*

(iii) *Suppose that* $(g^\square, r^\square) \neq (0,3), (1,1)$. *Write* $\iota_\Pi^\square \colon \Pi^\square \to \Pi_1^\square \times \cdots \times \Pi_1^\square$ *for the outer homomorphism induced by* $\iota^\square \colon X_{n^\square}^{\log \square} \to X^{\log \square} \times_{S^\square} \cdots \times_{S^\square} X^{\log \square}$ *(cf. Definition 2.2, (viii)), where* $\Pi_1^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l}(X^{\log \square})$ *(for a suitable choice of basepoint). Then* ϕ *induces a commutative diagram*

$$\begin{tikzcd} \Pi^\circ \arrow[r, "\phi"] \arrow[d, "\iota_\Pi^\circ"'] & \Pi^\bullet \arrow[d, "\iota_\Pi^\bullet"] \\ \Pi_1^\circ \times \cdots \times \Pi_1^\circ \arrow[r, dotted, "\sim"] & \Pi_1^\bullet \times \cdots \times \Pi_1^\bullet, \end{tikzcd}$$

where the lower horizontal isomorphism preserves the respective direct product decompositions (but possibly permutes the factors).

Proof. Assertion (i) follows from [HMM], Theorem A, (i). Assertion (ii) follows from [MzTa], Corollary 6.3. Assertion (iii) follows from assertion (ii). $\square$

4. Reconstruction of non-degenerate elements of log-full subgroups

We continue with the notation of §3. In the present §4, we reconstruct the subset of scheme-theoretically non-degenerate elements (cf. Definition 4.6, (i), below) of a log-full subgroup (cf. Theorem 4.15 below).

Proposition 4.1. *Let* $m < n$ *be an integer,* $q \colon X_n^{\log} \to X_m^{\log}$ *a projection,* V *a log divisor of* $X_n^{\log}$. *Write* $q \colon \Pi_n \to \Pi_m$ *for the outer homomorphism induced by* $q \colon X_n^{\log} \to X_m^{\log}$. *Suppose that* $q(V) \subsetneq X_m$. *Then the following hold:*

(i) $q(V)$ *is a log divisor of* $X_m^{\log}$.

(ii) *Let* $I_V \subseteq \Pi_n$ *be an inertia group associated to* V. *Then* $q(I_V)$ $(\simeq I_V)$ *is an inertia group associated to* $q(V)$.

Proof. Assertion (i) follows immediately from the latter portion of Definition 2.2, (vi), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ and $X_m^{\log}$. Assertion (ii) follows from [NodNon], Remark 2.4.2, together with the *surjectivity* portion of [NodNon], Lemma 2.7, (ii). $\square$

Proposition 4.2. *Let* P *be a log-full point of* $X_n^{\log}$; $V_1, \ldots, V_n$ *the log divisors such that* $P = V_1 \cap \cdots \cap V_n$; $A = I_{V_1} \times \cdots \times I_{V_n}$ *the log-full subgroup at* P *(cf. Proposition 3.7, (iii), (iv)). Then the following hold:*

(i) *There exists a tripodal divisor in* $\{V_1, \ldots, V_n\}$. *Suppose that* V_1 *is a tripodal divisor. Thus,* $\mathcal{G}_{V_1}$ *has precisely two vertices* v_1, v_1', *one of which is a tripod. Suppose that* v_1 *is a tripod.*

(ii) *If* $r = 1$, *then there exists a unique* (g,r)*-divisor in* $\{V_1, \ldots, V_n\}$. *Suppose that* V_n *is this unique* (g,r)*-divisor.*

(iii) *In the situation of (i), if* $(g,r) \neq (0,3)$, *then there exists an* $i_0 \in \{1+r, \ldots, n+r\}$ *such that* c_{i_0} *is a cusp of* $\mathcal{G}_{V_1}|_{v_1}$ *(cf.* [CbTpI], *Definition 2.1, (iii)). In this case, write* $p \colon X_n^{\log} \to X_{n-1}^{\log}$ *for the projection morphism of profile* $\{i_0 - r\}$ *(cf.* [MzTa], *Definition 2.1, (ii)).*

(iv) *In the situation of (i), if* $(g,r) = (0,3)$, *then there exists an* $i_0 \in \{1, \ldots, 3+n\}$ *such that* c_{i_0} *is a cusp of* $\mathcal{G}_{V_1}|_{v_1}$. *In this case, write* $p \colon X_n^{\log} \to X_{n-1}^{\log}$ *for the* *morphism determined by the morphism* $(\overline{\mathcal{M}}^{\log}_{0,n+3})_k \to (\overline{\mathcal{M}}^{\log}_{0,n+2})_k$ *obtained by forgetting the* i_0*-th marked point (cf. the proof of Proposition 3.4, (iii))*.

(v) In the situation of (iii) or (iv), it holds that $V'_1 \overset{\mathrm{def}}{=} p(V_1) = X_{n-1}$ *and* $V'_i \overset{\mathrm{def}}{=} p(V_i)$ *is a log divisor of* $X^{\log}_{n-1}$*, for all* $i \in \{2, \ldots, n\}$.

(vi) In the situation of (v), it holds that $V'_i \neq V'_j$*, for all* $i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}$.

(vii) In the situation of (v), it holds that $p(P)$ *is a log-full point of* $X^{\log}_{n-1}$.

(viii) In the situation of (iii) or (iv), for arbitrary (g, r)*, we write, by abuse of notation,* $p\colon \Pi_n \to \Pi_{n-1}$ *for the (outer) homomorphism induced by* p*. Then* $A' \overset{\mathrm{def}}{=} p(A)$ *is a log-full subgroup of* Π_{n-1}*, and we have exact sequences*

$$1 \longrightarrow \Pi_{n/n-1} \overset{\mathrm{def}}{=} \mathrm{Ker}(p) \longrightarrow \Pi_n \overset{p}{\longrightarrow} \Pi_{n-1} \longrightarrow 1,$$

$$1 \longrightarrow I_{V_1} \longrightarrow A \overset{p}{\longrightarrow} A' \longrightarrow 1.$$

Proof. Assertions (i), (ii) follow from Proposition 3.7, (i), (ii) (cf. also Lemma 3.9). Assertion (iii) follows from Proposition 3.3, (ii). Assertion (iv) is immediate. Assertion (v) follows from our choice of $p\colon X^{\log}_n \to X^{\log}_{n-1}$, together with the *terminality* of v_1 (cf. also Proposition 4.1, (i)). Next, we verify assertion (vi). By assertion (v), it holds that $V'_1 \neq V'_j$, for all $j \in \{2, \ldots, n\}$. Thus, we may assume without loss of generality that $1 < i < j$, and that $\mathcal{G}_{V_i}$ has precisely two vertices v_i, w_i such that c_{i_0} is a cusp of $\mathcal{G}_{V_i}|_{v_i}$. Let us recall that we have identified $\mathrm{Cusp}(\mathcal{G}_{V_i})$, $\mathrm{Cusp}(\mathcal{G}_{V_j})$ with $C_{r,n}$ (cf. Definition 2.2, (v)). Suppose that $V'_i = V'_j$. Observe that c_{i_0} does not belong to the set of cusps of any tripod (vertex) of $\mathcal{G}_{V_i}, \mathcal{G}_{V_j}$. Thus, one verifies easily that $\mathcal{G}_{V_j}$ has precisely two vertices v_j, w_j such that

$$(\mathrm{Cusp}(\mathcal{G}_{V_j}|_{v_j}) \cap \mathrm{Cusp}(\mathcal{G}_{V_j})) \cup \{c_{i_0}\} = \mathrm{Cusp}(\mathcal{G}_{V_i}|_{v_i}) \cap \mathrm{Cusp}(\mathcal{G}_{V_i});$$

$$\sharp\mathrm{Cusp}(\mathcal{G}_{V_j}|_{v_j}) + 1 = \sharp\mathrm{Cusp}(\mathcal{G}_{V_i}|_{v_i});$$

$$(\mathrm{Cusp}(\mathcal{G}_{V_i}|_{w_i}) \cap \mathrm{Cusp}(\mathcal{G}_{V_i})) \cup \{c_{i_0}\} = \mathrm{Cusp}(\mathcal{G}_{V_j}|_{w_j}) \cap \mathrm{Cusp}(\mathcal{G}_{V_j});$$

$$\sharp\mathrm{Cusp}(\mathcal{G}_{V_i}|_{w_i}) + 1 = \sharp\mathrm{Cusp}(\mathcal{G}_{V_j}|_{w_j});$$

$$g(v_i) = g(v_j),\ g(w_i) = g(w_j),$$

where we write $g(v_{(-)}), g(w_{(-)})$ for the "genus" of $\mathcal{G}_{V_{(-)}}|_{v_{(-)}}, \mathcal{G}_{V_{(-)}}|_{w_{(-)}}$ (cf. [CbTpI], Definition 2.3, (ii)). But one verifies easily from the correspondence between *log divisors* and *subsets* of $C_{r,n}$ (cf. Remark 2.4), together with the definition of c_{i_0} in the statements of assertions (iii), (iv), that this implies that there exists a *tripod* (vertex) v_P of $\mathcal{G}_P$ such that

$$\mathrm{Cusp}(\mathcal{G}_P|_{v_P}) \cap \mathrm{Cusp}(\mathcal{G}_P) = \{c_{i_0}\}.$$

On the other hand, this contradicts the *terminality* of the tripodal divisor V_1 (cf. Lemma 3.9). In particular, we conclude that $V'_i \neq V'_j$. Assertion (vii) follows from assertion (vi). Finally, assertion (viii) follows from assertions (v), (vii), together with [MzTa], Proposition 2.2, (i). □

Proposition 4.3. *Let P be a log-full point of $X_n^{\log}$ and I_V an inertia group associated to a log divisor V. Then it holds that*

$$P \in V \iff \text{there exists a log-full subgroup } A \text{ at } P \text{ such that } I_V \subseteq A.$$

Proof. The implication $\Longrightarrow$ follows immediately from Proposition 3.7, (iii), (iv). Thus, it suffices to consider the implication $\Longleftarrow$. Let $V_1, \ldots, V_n$ be log divisors such that $P = V_1 \cap \cdots \cap V_n$ (cf. Proposition 3.7, (iii), (iv)). We may assume without loss of generality that V_1 is a tripodal divisor (cf. Proposition 4.2, (i)). In the following, we consider a projection $p\colon X_n^{\log} \to X_{n-1}^{\log}$ as in Proposition 4.2, (iii) or (iv), and the corresponding (outer) homomorphism $p\colon \Pi_n \to \Pi_{n-1}$ of Proposition 4.2, (viii).

Let us first suppose that $p(V) = X_{n-1}$. Then since the generic point of V maps via p to the generic point of X_{n-1}, $I_V \subseteq \mathrm{Ker}(p) \cap A = I_{V_1}$ (cf. Proposition 4.2, (viii)). Now observe that since $p(V) = p(V_1) = X_{n-1}$, I_V and I_{V_1} may be regarded as cuspidal inertia groups of the smooth log curve determined by the geometric generic fiber of $p\colon X_n^{\log} \to X_{n-1}^{\log}$. In particular, the inclusion $I_V \subseteq I_{V_1}$ of profnite groups isomorphic to $\mathbb{Z}_l$ (cf. Proposition 3.7, (iii)) implies, by [CmbGC], Proposition 1.2, (i), that $P \in V_1 = V$.

Thus, it suffices to consider the case where $p(V) \neq X_{n-1}$. Then by Proposition 4.1, (i), (ii), $V' \stackrel{\mathrm{def}}{=} p(V)$ is a log divisor of $X_{n-1}^{\log}$, and p induces an isomorphism $I_V \xrightarrow{\sim} I_{V'}$. Now we apply induction on n. Here, we note that although we have assumed that $n > 1$, the assertion corresponding to the implication $\Longleftarrow$ for $n = 1$ follows immediately from [CmbGC], Proposition 1.2, (i). Since $A' \stackrel{\mathrm{def}}{=} p(A)$ is a log-full subgroup at $P' \stackrel{\mathrm{def}}{=} p(P)$ (cf. Proposition 4.2, (viii)) that contains $I_{V'}$, it follows from the induction hypothesis that $P' = V_2' \cap \cdots \cap V_n' \in V'$, where $V_i' \stackrel{\mathrm{def}}{=} p(V_i)$ and $i \in \{2, \ldots, n\}$. By Proposition 3.7, (iv), it holds that $V' \in \{V_2', \ldots, V_n'\}$, so we may assume without loss of generality that $V' = V_2'$. It follows immediately from the latter portion of Definition 2.2, (vi), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ and $X_{n-1}^{\log}$, that there exists a log divisor $W \neq V_2$ of $X_n^{\log}$ such that $p(W) = V_2'$ and $(V \subseteq)\ p^{-1}(V_2') = V_2 \cup W$. Note that $V \in \{V_2, W\}$. Suppose that $V = W$. Then

$$I_W = I_V \subseteq p^{-1}p(I_V) = p^{-1}(I_{V'}) = p^{-1}(I_{V_2'}) = I_{V_1} \oplus I_{V_2},$$

where the last equality follows from Proposition 4.2, (viii). Now let us consider the stable log curve obtained by restricting $p\colon X_n^{\log} \to X_{n-1}^{\log}$ to the generic point of V_2'. This stable log curve has precisely two irreducible components, corresponding to V_2 and W, whose intersection consists of precisely one node, which we denote by $e_{V_2 \cap W}$; moreover, V_1 may be regarded as a cusp of this stable log curve which, since $V_1 \cap V_2 \neq \emptyset$, is contained in the irreducible component of the stable log curve corresponding to V_2. Write $I_{V_2 \cap W}$ for the inertia group of $e_{V_2 \cap W}$ such that

$$I_{V_2 \cap W} \subseteq I_{V_2} \oplus I_W = I_{V_2} \oplus I_V.$$

Since $(I_V \subset)\ I_{V_1} \oplus I_{V_2}$ is an abelian group, it holds that (the cuspidal inertia group) I_{V_1} commutes with I_{V_2} and $I_V = I_W$, hence with (the nodal inertia group) $I_{V_2 \cap W}$. In particular, by [CmbGC], Proposition 1.2, (i), (ii), we obtain a contradiction to our assumption that $V = W$. Thus, $V = V_2$, and $P \in V_2 = V$. $\square$

Proposition 4.4. *Let V, W be log divisors and I_V an inertia group associated to V. Then it holds that*

$$W = V \Longleftrightarrow \text{ there exists an inertia group } I_W \text{ associated to } W \text{ such that } I_W = I_V.$$

Proof. The implication $\Longrightarrow$ follows immediately from the various definitions involved. Thus, it suffices to consider the implication $\Longleftarrow$. Recall that it follows from the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ that there exists a log-full point P such that $P \in V$. Let $V_1, \ldots, V_n$ be the distinct log divisors such that $P = V_1 \cap \cdots \cap V_n$, $V \in \{V_1, \ldots, V_n\}$ (cf. Proposition 3.7, (iii), (iv)). Thus, we may assume without loss of generality that $I_V = I_{V_i} \subseteq A$ for some $i \in \{1, \ldots, n\}$. In the following, we assume that there exists an inertia group I_W associated to the log divisor W such that $I_W = I_V$ and consider a projection $p \colon X_n^{\log} \to X_{n-1}^{\log}$ as in Proposition 4.2, (iii) or (iv), and the corresponding (outer) homomorphism $p \colon \Pi_n \to \Pi_{n-1}$ of Proposition 4.2, (viii).

Let us first suppose that $p(V) = X_{n-1}$. Then since the generic point of V maps via p to the generic point of X_{n-1}, $I_V \subseteq \mathrm{Ker}(p) \cap A = I_{V_1}$ (cf. Proposition 4.2, (viii)). Since $I_W = I_V \subseteq \mathrm{Ker}(p)$, it follows from Propositions 3.7, (iii); 4.1, (i), (ii), that $p(W) = X_{n-1}$. Now observe that since $p(V) = p(W) = p(V_1) = X_{n-1}$, I_V, I_W, and I_{V_1} may be regarded as cuspidal inertia groups of the smooth log curve determined by the geometric generic fiber of $p \colon X_n^{\log} \to X_{n-1}^{\log}$. In particular, the equality and inclusion $I_W = I_V \subseteq I_{V_1}$ of profinite groups isomorphic to $\mathbb{Z}_l$ (cf. Proposition 3.7, (iii)) implies, by [CmbGC], Proposition 1.2, (i), that $W = V = V_1$.

Thus, it suffices to consider the case where $p(V) \neq X_{n-1}$. By Proposition 4.1, (i), (ii), $V' \stackrel{\mathrm{def}}{=} p(V)$ is a log divisor of $X_{n-1}^{\log}$, and p induces an isomorphism $I_V \xrightarrow{\sim} I_{V'}$. If $p(W) = X_{n-1}$, then since the generic point of W maps via p to the generic point of X_{n-1}, it follows that $I_W = I_V \subseteq \mathrm{Ker}(p)$, in contradiction to the existence of the isomorphism $I_V \xrightarrow{\sim} I_{V'}$ (cf. Proposition 3.7, (iii)). Thus, we conclude that $p(W) \neq X_{n-1}$ and hence, by Proposition 4.1, (i), (ii), that $W' \stackrel{\mathrm{def}}{=} p(W)$ is a log divisor of $X_{n-1}^{\log}$, and $I_W \xrightarrow{\sim} I_{W'}$. Now we apply induction on n. Here, we note that although we have assumed that $n > 1$, the assertion corresponding to the implication $\Longleftarrow$ for $n = 1$ follows immediately from [CmbGC], Proposition 1.2, (i). Then since $I_{W'} = I_{V'}$, it follows from the induction hypothesis that $W' = V'$. Now suppose that $W \neq V$. Then it follows immediately from the latter portion of Definition 2.2, (vi), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ and $X_{n-1}^{\log}$, that the stable log curve obtained by restricting $p \colon X_n^{\log} \to X_{n-1}^{\log}$ to the generic point of $W' = V'$ has precisely two irreducible components, corresponding to W and V, whose intersection consists of precisely one node. Thus, since $I_W = I_V$, we conclude from [CmbGC], Proposition 1.2, (i), that $W = V$. $\square$

Proposition 4.5. *Let $P^\dagger, P^\ddagger$ be log-full points of $X_n^{\log}$ and $A^\dagger$ a log-full subgroup at $P^\dagger$. Then it holds that*

$$P^\dagger = P^\ddagger \Longleftrightarrow \text{ there exists a log-full subgroup } A^\ddagger \text{ at } P^\ddagger \text{ such that } A^\dagger = A^\ddagger.$$

In particular, the assignment $P^\dagger \mapsto [A^\dagger]$ (where "$[(-)]$" denotes the Π_n-conjugacy class of "$(-)$") determines a natural bijection

$$\{\text{log-full points}\} \xrightarrow{\sim} \{\Pi_n\text{-conjugacy classes of log-full subgroups}\}.$$

Proof. The assertion of the second display follows from the assertion of the first display. Let us prove the assertion of the first display. The implication $\Longrightarrow$ is immediate. Thus, it suffices to prove the implication $\Longleftarrow$. Suppose that $A^\dagger = A^\ddagger$. Let $V_1, \ldots, V_n$ be log divisors such that $P^\dagger = V_1 \cap \cdots \cap V_n$; write $A^\dagger = I_{V_1} \times \cdots \times I_{V_n}$ (cf. Proposition 3.7, (iii), (iv)). In particular, for each $j \in \{1, \ldots, n\}$, $I_{V_j} \subseteq A^\dagger = A^\ddagger$. In particular, it follows from Proposition 4.3 that $P^\ddagger \in V_j$. Thus, $P^\ddagger \in V_1 \cap \cdots \cap V_n = P^\dagger$ (cf. the notational conventions of Definition 2.2, (ii)), i.e., $P^\dagger = P^\ddagger$, as desired. □

In the remainder of the present §4, we shall apply the notational conventions introduced in the statement of Proposition 4.2 (cf, especially, Proposition 4.2, (i), (ii)).

Definition 4.6. Let $\alpha \in A$ and

$$A = I_{V_1} \times \cdots \times I_{V_n} : \alpha \mapsto (a_1, \ldots, a_n).$$

(i) We shall say that α is *scheme-theoretically non-degenerate* if $a_i \neq 1_A$ for each $i \in \{1, \ldots, n\}$.

(ii) We shall say that α is *group-theoretically non-degenerate* if $Z_{\Pi_n}(\alpha)$ is an abelian group.

Theorem 4.7. *It holds that*

$$\{\textit{scheme-theoretically non-degenerate elements of } A\}$$

$$= \{\textit{group-theoretically non-degenerate elements of } A\}.$$

Proof. When $r \neq 1$, this follows from Propositions 4.9, 4.12, below. When $r = 1$, this follows from Propositions 4.9, 4.12, and 4.14, below. □

Lemma 4.8. *It holds that*

$$N_{\Pi_n}(A) = A,$$

i.e., every log-full subgroup of Π_n *is normally terminal in* Π_n*.*

Proof. In the following, we consider the projection $p\colon X_n^{\log} \to X_{n-1}^{\log}$ of Proposition 4.2, (iii) or (iv), and the associated (outer) homomorphism $p\colon \Pi_n \to \Pi_{n-1}$ of Proposition 4.2, (viii).

We apply induction on n. Here, we note that although we have assumed that $n > 1$, the analogous assertion for $n = 1$ follows immediately from [CmbGC], Proposition 1.2, (ii). By definition, $N_{\Pi_n}(A) \supseteq A$. Let $\alpha \in N_{\Pi_n}(A)$. Since $\alpha A \alpha^{-1} = A$, it follows that $p(\alpha) A' p(\alpha)^{-1} = A'$, where we recall the log-full subgroup $A' = p(A)$ of Π_{n-1} discussed in Proposition 4.2, (viii). Then it follows from the induction hypothesis that A' is normally terminal. Thus, $p(\alpha) \in A'$, i.e., $p(N_{\Pi_n}(A)) \subseteq A'$. Since $p(N_{\Pi_n}(A)) \supseteq p(A) = A'$, it follows that $p(N_{\Pi_n}(A)) = A'$.

Next, we observe that by Proposition 4.2, (viii), $N_{\Pi_n}(A) \cap \Pi_{n/n-1} \supseteq A \cap \Pi_{n/n-1} = I_{V_1}$. Let $\alpha \in N_{\Pi_n}(A) \cap \Pi_{n/n-1}$. Since $\alpha A \alpha^{-1} = A$, and $\Pi_{n/n-1}$ is normal in Π_n (cf. Proposition 4.2, (viii)), it follows that $\alpha I_{V_1} \alpha^{-1} = I_{V_1} \subseteq A$. On the other hand, let us observe that V_1 determines a cusp of the smooth log curve obtained by restricting $p\colon X_n^{\log} \to X_{n-1}^{\log}$ to the generic point of X_{n-1} (cf. Proposition 4.2, (v)). Thus, we conclude from [CmbGC], Proposition 1.2, (ii), that $\alpha \in N_{\Pi_{n/n-1}}(I_{V_1}) = I_{V_1}$, i.e., that $N_{\Pi_n}(A) \cap \Pi_{n/n-1} = I_{V_1}$.

It follows from the above discussion that we have an exact sequence

$$1 \longrightarrow I_{V_1} \longrightarrow N_{\Pi_n}(A) \xrightarrow{p} A' \longrightarrow 1.$$

By the five lemma (cf. Proposition 4.2, (viii)), it thus follows that $N_{\Pi_n}(A) = A$. □

Proposition 4.9. *Let $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. If $a_1, \ldots, a_n \neq 1_A$, then $Z_{\Pi_n}(a_1 \cdots a_n) = A$, hence, in particular, is an abelian group.*

Proof. Let $X^{\log}_{n+1} \to X^{\log}_n$ be the projection morphism of profile $\{n+1\}$. This projection induces an exact sequence

$$1 \longrightarrow \Pi_{n+1/n} \longrightarrow \Pi_{n+1} \longrightarrow \Pi_n \longrightarrow 1,$$

which gives rise to an outer representation $\rho\colon \Pi_n \to \mathrm{Out}(\Pi_{n+1/n})$. Recall that ρ is injective (cf. [Asd], the Remark following the proof of Theorem 1). Moreover, recall that there exists an isomorphism $\Pi_{\mathcal{G}_P} \xrightarrow{\sim} \Pi_{n+1/n}$ such that ρ determines an isomorphism

$$A \xrightarrow{\sim} \mathrm{Dehn}(\mathcal{G}_P)$$

(cf. [CbTpI], Definition 4.4; [CbTpI], Proposition 5.6, (ii)), and, moreover, it holds that

$$\mathrm{Aut}(\mathcal{G}_P) = N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{n+1/n})}(\mathrm{Dehn}(\mathcal{G}_P))$$

(cf. [CbTpI], Theorem 5.14, (iii)).

Since $A \simeq \mathbb{Z}_l^{\oplus n}$ (cf. Proposition 3.7, (iii)) is an abelian group, it suffices to verify that $Z_{\Pi_n}(a_1 \cdots a_n) = A$. Since A is an abelian group, and $a_1 \cdots a_n \in A \subseteq \Pi_n$, it follows that $Z_{\Pi_n}(a_1 \cdots a_n) \supseteq A$. By [NodNon], Theorem A (cf. also [NodNon], Remark 2.4.2), and [CbTpI], Corollary 5.9, (ii), it follows that $\rho(Z_{\Pi_n}(a_1 \cdots a_n)) \subseteq \mathrm{Aut}(\mathcal{G}_P)$. Thus, we conclude that

$$\begin{aligned}\rho(Z_{\Pi_n}(a_1 \cdots a_n)) &\subseteq \mathrm{Aut}(\mathcal{G}_P) \cap \rho(\Pi_n) = N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{n+1/n})}(\mathrm{Dehn}(\mathcal{G}_P)) \cap \rho(\Pi_n) \\ &= N_{\rho(\Pi_n)}(\mathrm{Dehn}(\mathcal{G}_P)) = N_{\rho(\Pi_n)}(\rho(A)) = \rho(N_{\Pi_n}(A)).\end{aligned}$$

In particular, $Z_{\Pi_n}(a_1 \cdots a_n) \subseteq N_{\Pi_n}(A) = A$ (cf. Lemma 4.8). □

Definition 4.10. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-l PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$. Suppose that $\mathbb{G}$ is a tree. Let $e_1, e_2 \in \mathrm{Edge}(\mathcal{G})$; b_1, b'_1 the two branches of e_1; b_2, b'_2 the two branches of e_2. We suppose that $\mathbb{G}_{\not\ni b_1} \cap \mathbb{G}_{\not\ni b_2} = \emptyset$ (cf. Definition 2.5). Write $\mathbb{H}$ for the semi-graph obtained by considering the "intersection" (in the evident sense) of $\mathbb{G}_{\ni b_1}$ and $\mathbb{G}_{\ni b_2}$. Then we define the semi-graph of anabelioids of pro-l PSC-type

$$\mathcal{G}_{b_1 \curlyvee b_2}$$

(obtained by *"switching"* b_1 and b_2) as follows. We take the underlying semi-graph $\mathbb{G}_{b_1 \curlyvee b_2}$ of $\mathcal{G}_{b_1 \curlyvee b_2}$ to be the semi-graph obtained by "gluing" $\mathbb{H}$ to $\mathbb{G}_{\not\ni b_1}$ and $\mathbb{G}_{\not\ni b_2}$ in the following way:

- we glue the branch of $\mathbb{H}$ corresponding to b_1 and the branch of $\mathbb{G}_{\not\ni b_2}$ corresponding to b'_2 along a *single edge* (whose branches correspond to the two branches that are glued to one another);
- we glue the branch of $\mathbb{H}$ corresponding to b_2 and the branch of $\mathbb{G}_{\not\ni b_1}$ corresponding to b'_1 along a *single edge* (whose branches correspond to the two branches that are glued to one another).

Then the various connected anabelioids that constitute $\mathcal{G}$ naturally determine a semi-graph of anabelioids of pro-l PSC-type $\mathcal{G}_{b_1 \curlyvee b_2}$ whose underlying semi-graph is the semi-graph $\mathbb{G}_{b_1 \curlyvee b_2}$.

Proposition 4.11. *Suppose that $r \neq 1$ (resp. $r = 1$). Let $i \in \{1, \ldots, n\}$ (resp. $i \in \{1, \ldots, n-1\}$). Then there exists a log divisor $H \neq V_i$ such that*

$$V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$$

is a log-full point ($\neq P$).

Proof. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G} \stackrel{\mathrm{def}}{=} \mathcal{G}_P$. It follows from Proposition 3.7, (ii), that there exists a node $e \in \mathrm{Node}(\mathcal{G})$ such that V_i is the log divisor associated to $e \in \mathrm{Node}(\mathcal{G})$. Let $w_1, w_2 \in \mathrm{Vert}(\mathcal{G})$ be distinct vertices such that e abuts to w_1, w_2.

First, let us suppose that w_1, w_2 are tripods. Then let us observe that there exist distinct elements

$$y_1, z_1, y_2, z_2 \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G})) \setminus \{e\}$$

such that (suitable branches of) e, y_1, z_1 give rise to the three cusps of $\mathcal{G}|_{w_1}$, and (suitable branches of) e, y_2, z_2 give rise to the three cusps of $\mathcal{G}|_{w_2}$.

Let b_1 be the branch of y_1 that abuts to w_1; b_2 the branch of y_2 that abuts to w_2; $\mathcal{G}' \stackrel{\mathrm{def}}{=} (\mathcal{G})_{b_1 \curlyvee b_2}$ (cf. Definition 4.10). Then it follows immediately from the definitions (Definitions 2.3, 4.10), together with the fact that

$$(\mathrm{Cusp}(\mathbb{G}_{\ni b_1}) \cap \mathrm{Cusp}(\mathbb{G}_{\ni b_2}) \cap C_{r,n}) \subsetneq C_{r,n}$$

(cf. Definition 2.3, Remark 2.4), that there exists a log divisor $H \neq V_i$ such that H is the log divisor associated to the element $e' \in \mathrm{Node}(\mathcal{G}')$ corresponding to $e \in \mathrm{Node}(\mathcal{G})$ and $V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point $P' \neq P$ such that $\mathcal{G}_{P'} = \mathcal{G}'$. (Here, we observe that for $j \in \{1, \ldots, n\} \setminus \{i\}$, V_j may be regarded as the log divisor associated to a suitable choice of element $e'_j \in \mathrm{Node}(\mathcal{G}')$ corresponding to the element $e_j \in \mathrm{Node}(\mathcal{G})$ to which the log divisor V_j is associated.) This completes the proof of Proposition 4.11 in the case where w_1, w_2 are tripods.

Thus, we may assume without loss of generality that w_2 is *not* a tripod. Then it follows from Proposition 3.7, (i), that w_1 is a tripod, and w_2 is of type $(g, r) \neq (0, 3)$. Next, let us observe that $r \neq 1$. Indeed, if $r = 1$, then it follows immediately from the fact that w_2 is of type $(g, r) \neq (0, 3)$, together with the definition of V_n (cf. Proposition 4.2, (ii)), that $V_i = V_n$. This contradicts our assumption that $i \leq n-1$ if $r = 1$. Thus, in summary, we may assume that w_1 is a tripod, w_2 is of type $(g, r) \neq (0, 3)$, and $r \neq 1$.

Next, let us observe that there exist distinct elements

$$y^\dagger, y^\ddagger, y^*, y_1, \ldots, y_{r-2} \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G})) \setminus \{e\}$$

such that (suitable branches of) $e, y^\dagger, y^\ddagger$ give rise to the three cusps of $\mathcal{G}|_{w_1}$, and (suitable branches of) $e, y^*, y_1, \ldots, y_{r-2}$ give rise to the r cusps of $\mathcal{G}|_{w_2}$. (Here, we remark that since $r \neq 0, 1$, it follows that $r + 1 \geq 3$.)

Let $b^\dagger$ be the branch of $y^\dagger$ that abuts to w_1; $b^\ddagger$ the branch of $y^\ddagger$ that abuts to w_1; b^* the branch of y^* that abuts to w_2. Then observe that, after possibly permuting the superscripts "†" and "‡", we may assume that $\mathrm{Cusp}((\mathbb{G})_{\ni b^\ddagger}) \supseteq \{c_1, \ldots, c_r\}$.

Let $\mathcal{G}' \stackrel{\mathrm{def}}{=} (\mathcal{G})_{b^\dagger \curlyvee b^*}$. Then it follows immediately from the definitions (cf. the choice of b_1; Definitions 2.3, 4.10), together with the fact that

$$(\mathrm{Cusp}(\mathbb{G}_{\ni b^\dagger}) \cap \mathrm{Cusp}(\mathbb{G}_{\ni b^*}) \cap C_{r,n}) \subsetneq C_{r,n}$$

(cf. Definition 2.3, Remark 2.4), that there exists a log divisor $H \neq V_i$ such that H is the log divisor associated to the element $e' \in \mathrm{Node}(\mathcal{G}')$ corresponding to $e \in \mathrm{Node}(\mathcal{G})$ and $V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point $P' \neq P$ such that $\mathcal{G}_{P'} = \mathcal{G}'$. (Here, we observe that for $j \in \{1, \ldots, n\} \setminus \{i\}$, V_j may be regarded as the log divisor associated to a suitable choice of element $e'_j \in \mathrm{Node}(\mathcal{G}')$ corresponding to the element $e_j \in \mathrm{Node}(\mathcal{G})$ to which the log divisor V_j is associated.) $\square$

Proposition 4.12. *Suppose that $r \neq 1$ (resp. $r = 1$). Let $i \in \{1, \ldots, n\}$ (resp. $i \in \{1, \ldots, n-1\}$) and $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. Then $Z_{\Pi_n}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)$ is a non-abelian group.*

Proof. By Proposition 4.11, there exists a log divisor $H \neq V_i$ such that $P' = V_1 \cap \cdots \cap V_{i-1} \cap H \cap V_{i+1} \cap \cdots \cap V_n$ is a log-full point. Write $A' = I_{V_1} \times \cdots \times I_{V_{i-1}} \times I_H \times I_{V_{i+1}} \times \cdots \times I_{V_n}$. Since

$$a_1 \cdots a_{i-1} a_{i+1} \cdots a_n \in A \cap A',$$

and A, A' are abelian groups, it follows that

$$A, A' \subseteq Z_{\Pi_n}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n).$$

Since A, A' are distinct log-full subgroups (cf. Proposition 4.5) and contained in $Z_{\Pi_n}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)$, by Lemma 4.8, it follows that $Z_{\Pi_n}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)$ is a non-abelian group. $\square$

Proposition 4.13. *If $r = 1$, then there exists an element $i \in \{1, \ldots, n\}$ such that the projection morphism $q \colon X_n^{\log} \to X^{\log}$ of co-profile $\{i\}$ (cf.* [MzTa]*, Definition 2.1, (ii)) induces an isomorphism $V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$.*

Proof. Let w be the unique vertex of $\mathcal{G}_P$ of genus g (cf. Proposition 3.7, (i)). (Note that since $r = 1$, it holds that $g \neq 0$.) Then since $r = 1$, it follows immediately from Propositions 3.7, (i); 4.2, (ii), that there exist a unique vertex $u \in \mathrm{Vert}(\mathcal{G}_P)$ and a unique node $e \in \mathrm{Node}(\mathcal{G}_P)$ (corresponding to V_n) such that e abuts to w, u, and, moreover, u is a tripod. Next, let us observe that there exist distinct elements $y^\dagger, y^\ddagger \in (C_{r,n} \coprod \mathrm{Node}(\mathcal{G}_P)) \setminus \{e\}$ such that (suitable branches of) $e, y^\dagger, y^\ddagger$ give rise to the three cusps of $\mathcal{G}_P|_u$. Let $b^\dagger$ be the branch of $y^\dagger$ that abuts to u, $b^\ddagger$ the branch of $y^\ddagger$ that abuts to u. Write $\mathbb{G}_P$ for the underlying semi-graph of $\mathcal{G}_P$. Thus, $y^\dagger \in \mathrm{Edge}((\mathbb{G}_P)_{\ni b^\ddagger})$, $y^\ddagger \in \mathrm{Edge}((\mathbb{G}_P)_{\ni b^\dagger})$. Then observe that, after possibly permuting the superscripts "$\dagger$" and "$\ddagger$", we may assume that $c_1 \in \mathrm{Cusp}((\mathbb{G}_P)_{\ni b^\ddagger}) \setminus \{y^\ddagger\}$.

Note that since, whenever $y^\ddagger \notin C_{r,n}$, the genus portion of the type (i.e., "(g, r)") of the semi-graph of anabelioids of PSC-type $(\mathcal{G}_P)_{\not\ni b^\ddagger}$ is $= 0$ (cf. Proposition 3.7, (i)), the fact that $c_1 \in \mathrm{Cusp}((\mathbb{G}_P)_{\ni b^\ddagger}) \setminus \{y^\ddagger\}$ implies that

$$\text{either } y^\ddagger \in (\mathrm{Cusp}((\mathbb{G}_P)_{\ni b^\dagger}) \cap C_{r,n}) \setminus \{c_1\} \text{ or } \mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b^\ddagger}) \cap C_{r,n} \neq \emptyset.$$

In particular, since $y^\dagger \neq y^\ddagger$ and, whenever $y^\ddagger \notin C_{r,n}$, $\mathrm{Cusp}((\mathbb{G}_P)_{\not\ni b^\ddagger}) \cap C_{r,n} \subseteq \mathrm{Cusp}((\mathbb{G}_P)_{\ni b^\dagger}) \setminus \{y^\dagger\}$, there exists an element $i \in \{1, \ldots, n\}$ such that $x_i \in \mathrm{Cusp}((\mathbb{G}_P)_{\ni b^\dagger}) \setminus \{y^\dagger\}$. Now it follows immediately from our choice of i, together with the fact that the divisor V_n corresponds to the node e (cf. Propositions 3.7, (ii);

4.2, (ii)), that the projection morphism $q\colon X_n^{\log} \to X^{\log}$ of co-profile $\{i\}$ induces an isomorphism $q\colon V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$, as desired. $\square$

Proposition 4.14. *Let* $(a_1, \ldots, a_n) \in I_{V_1} \times \cdots \times I_{V_n} = A$. *If* $r = 1$, *then* $Z_{\Pi_n}(a_1 \cdots a_{n-1})$ *is a non-abelian group.*

Proof. By Proposition 4.13, there exists an element $i \in \{1, \ldots n\}$ such that the projection morphism $q\colon X_n^{\log} \to X^{\log}$ of co-profile $\{i\}$ induces an isomorphism $V_1 \cap \cdots \cap V_{n-1} \xrightarrow{\sim} X$. By abuse of notation, we write $q\colon \Pi_n \to \Pi_1$ for the outer homomorphism induced by q. Write $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}$ for the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the reduced closed subscheme of X_n determined by $V_1 \cap \cdots \cap V_{n-1}$; $V_j^{\log}$, where $j \in \{1, \ldots, n-1\}$, for the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the reduced closed subscheme of X_n determined by V_j. Then it follows immediately that the morphism $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to V_j^{\log} \to X^{\log}$ induced by $q\colon X_n^{\log} \to X^{\log}$ determines (for suitable choices of basepoints) homomorphisms of profinite groups

$$\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \to \pi_1^{\text{pro-}l}(V_j^{\log}) \to \Pi_n \to \Pi_1.$$

Note that it follows immediately from the definition of I_{V_j} as an inertia group (cf. Proposition 3.7, (iii)) that, for suitable choices of basepoints in the $\pi_1(-)$'s of the above display, the image of $\pi_1^{\text{pro-}l}(V_j^{\log})$ in Π_n, hence also the image of $\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log})$ in Π_n, is contained in $Z_{\Pi_n}(I_{V_j}) \subset \Pi_n$. In particular, we obtain homomorphisms of profinite groups

$$\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \to D_{V_1} \cap \cdots \cap D_{V_{n-1}} \hookrightarrow \Pi_n \to \Pi_1,$$

where $D_{V_j} \stackrel{\text{def}}{=} Z_{\Pi_n}(I_{V_j})$ is the decomposition group associated to V_j determined by I_{V_j} (cf. [Hsh], Corollary 2). Next, observe that it follows from the well-known modular interpretation of the log moduli stacks involved (cf. Definition 2.2, (vi)) that $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to X^{\log}$ is *of type* $\mathbb{N}^{\oplus n-1}$ (cf. [Hsh], Definition 6). Since $V_1^{\log} \cap \cdots \cap V_{n-1}^{\log} \to X^{\log}$ is of type $\mathbb{N}^{\oplus n-1}$, one verifies immediately that the composite $\pi_1^{\text{pro-}l}(V_1^{\log} \cap \cdots \cap V_{n-1}^{\log}) \to \Pi_1$ is a surjection. In particular, the composite $D_{V_1} \cap \cdots \cap D_{V_{n-1}} \hookrightarrow \Pi_n \to \Pi_1$ is a surjection, i.e., $q(D_{V_1} \cap \cdots \cap D_{V_{n-1}}) = \Pi_1$. Thus, it follows immediately from the definitions that

$$\begin{aligned}\Pi_1 &= q(D_{V_1} \cap \cdots \cap D_{V_{n-1}}) = q(Z_{\Pi_n}(I_{V_1}) \cap \cdots \cap Z_{\Pi_n}(I_{V_{n-1}})) \\ &\subseteq q(Z_{\Pi_n}(a_1) \cap \cdots \cap Z_{\Pi_n}(a_{n-1})) \subseteq q(Z_{\Pi_n}(a_1 \cdots a_{n-1})) \subseteq \Pi_1.\end{aligned}$$

In particular, $q(Z_{\Pi_n}(a_1 \cdots a_{n-1})) = \Pi_1$, hence also $Z_{\Pi_n}(a_1 \cdots a_{n-1})$, is a non-abelian group. $\square$

Theorem 4.15. *For* $\square \in \{\circ, \bullet\}$, *let* $l^\square$ *be a prime number;* $k^\square$ *an algebraically closed field of characteristic* $\neq l^\square$; $S^\square \stackrel{\text{def}}{=} \operatorname{Spec}(k^\square)$; $(g^\square, r^\square)$ *a pair of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$;

$$X^{\log\,\square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log\,\square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log\,\square} \to S^\square$; $\Pi^\square \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log\,\square})$ *(for a suitable choice of basepoint);*

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups; A° a log-full subgroup of Π°. We suppose that $r^\square > 0$, and that $A^\bullet \stackrel{\mathrm{def}}{=} \phi(A^\circ)$ is a log-full subgroup of $\Pi^\bullet$. Then ϕ induces a bijection between the set of scheme-theoretically non-degenerate elements (cf. Definition 4.6, (i)) of A° and the set of scheme-theoretically non-degenerate elements of $A^\bullet$.

Proof. This follows immediately from Theorem 4.7. $\square$

5. **Reconstruction of log divisors**

We continue with the notation of §4. In the present §5, we reconstruct the set of inertia groups associated to log divisors (cf. Theorem 5.2 below).

Lemma 5.1. *The following hold:*

(i) There exists a unique collection of subgroups $B_1^\dagger, \ldots, B_n^\dagger \subseteq A$ such that the following hold:
  - *(a) $\dim_{\mathbb{Q}_l}(B_i^\dagger \otimes \mathbb{Q}_l) = n-1$, for each $i \in \{1, \ldots, n\}$.*
  - *(b) For each $i \in \{1, \ldots, n\}$, no element of $B_i^\dagger$ is (group-theoretically) non-degenerate.*
  - *(c) $B_i^\dagger = A \cap (B_i^\dagger \otimes \mathbb{Q}_l) \subset A \otimes \mathbb{Q}_l$, for all $i \in \{1, \ldots, n\}$.*

(ii) In the situation of (i), $\{B_i^\dagger \mid i \in \{1, \ldots, n\}\} = \{B_j \stackrel{\mathrm{def}}{=} \prod_{m \in \{1,\ldots,n\}\setminus\{j\}} I_{V_m} \mid j \in \{1, \ldots, n\}\}$.

(iii) In the situation of (i), $\{I_{V_1}, \ldots, I_{V_n}\} = \{\bigcap_{m \in \{1,\ldots,n\}\setminus\{j\}} B_m^\dagger \mid j \in \{1, \ldots, n\}\}$.

Proof. For $a \in A$, we shall write

$$J(a) \stackrel{\mathrm{def}}{=} \{m \in \{1, \ldots, n\} \mid a \notin B_j\}.$$

Observe that if $a_1, a_2 \in A$ are such that $J(a_1), J(a_2) \neq \emptyset$, and $J(a_1) \cap J(a_2) = \emptyset$, then there exists an element $\lambda \in \mathbb{Z}_l$ such that $a_1 a_2^\lambda$ is non-degenerate. Now assertions (i), (ii) follow the definitions, together with this observation. Assertion (iii) follows immediately from assertion (ii). $\square$

Theorem 5.2. *For $\square \in \{\circ, \bullet\}$, let $l^\square$ be a prime number; $k^\square$ an algebraically closed field of characteristic $\neq l^\square$; $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X^{\log \square}_{n^\square}$ the $n^\square$-th log configuration space associated to $X^{\log \square} \to S^\square$; $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log \square}_{n^\square})$ (for a suitable choice of basepoint);

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$, and that ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of inertia groups of Π° associated to log divisors of $X^{\log \circ}_{n^\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to log divisors of $X^{\log \bullet}_{n^\bullet}$.

Proof. Recall that it follows from the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ that, for each log divisor $V^\dagger$ of $X_n^{\log}$, there exists a log-full point $P^\dagger$ such that $P^\dagger \in V^\dagger$. Thus, Theorem 5.2 follows from Proposition 3.7, (iii), (iv); Theorem 4.15; Lemma 5.1. $\square$

6. **Reconstruction of tripodal divisors**

We continue with the notation of §5. In the present §6, we reconstruct the set of inertia groups associated to tripodal divisors (cf. Theorem 6.6 below).

Lemma 6.1. *Let V be a log divisor of $X_n^{\log}$. Write $V^{\log}$ for the log scheme obtained by equipping V with the log structure induced by the log structure of $X_n^{\log}$. Let $Y^{\log} \to S$ be a smooth log curve of type $(0,3)$. For $m \in \mathbb{Z}_{>0}$, write $Y_m^{\log}$ for the m-th log configuration space associated to $Y^{\log} \to S$.*

(i) If V is a tripodal divisor, then $V^{\log \leq 1}$ is isomorphic to $U_{X_{n-1}}$.

(ii) If V is a (g,r)-divisor, then $V^{\log \leq 1}$ is isomorphic to $U_{Y_{n-1}}$.

(iii) If V is neither a tripodal divisor nor a (g,r)-divisor, then there exists an element $m \in \{1, \ldots, n-2\}$ such that $V^{\log \leq 1}$ is isomorphic to $U_{Y_m} \times_S U_{X_{n-1-m}}$.

Proof. These assertions follow immediately by considering the objects parametrized by the various schemes which appear in the assertions. □

Definition 6.2. We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \simeq G_1 \times G_2$, where G_1, G_2 are profinite groups, either G_1 or G_2 is the trivial group (cf. [Ind], Definition 1.1). We shall say that a profinite group G is *decomposable* if G is not indecomposable.

Remark 6.3. Let $m \in \mathbb{Z}_{>0}$. Then we recall from [Ind], Theorem 3.5 (cf. also [MzTa], Remark 1.2.2; [MzTa], Proposition 2.2, (i)), that Π_m is *indecomposable* and *nontrivial.* If, moreover $m > 1$, then (g,r,m) is completely determined by the isomorphism class of Π_m (cf. Theorem 3.10, (i)). If $m = 1$, then the isomorphism class of Π_m is completely determined by $2g-2+r$ (cf. [CmbGC], Remark 1.1.3; [MzTa], Remark 1.2.2).

Remark 6.4. Let V, $V^{\log}$ be as in Lemma 6.1; I_V an inertia group associated to V. Then we observe that, for suitable choices of basepoints, there is a natural homomorphism $\pi_1^{\text{pro-}l}(V^{\log}) \to Z_{\Pi_n}(I_V)$ (cf. [Hsh], Corollary 2). Moreover, this natural homomorphism is, in fact, *injective* (cf. (the evident pro-l version of) [SemiAn], Proposition 2.5, (i); [CmbGC], Remark 1.1.3; [MzTa], Proposition 2.2, (i); [AbsTpII], Remark 1.5.1) and *surjective* (cf. [AbsTpII], Remark 1.5.2; [AbsTpII], Proposition 1.6, (v)), hence yields an *isomorphism*

$$\pi_1^{\text{pro-}l}(V^{\log}) \xrightarrow{\sim} Z_{\Pi_n}(I_V).$$

Lemma 6.5. *Let V be a log divisor of $X_n^{\log}$ and I_V an inertia group associated to V. Then the following hold:*

(i) $Z_{\Pi_n}(I_V)/I_V$ is either decomposable, isomorphic to Π_{n-1}, or (in the notation of Lemma 6.1) isomorphic to $\Pi_{n-1}^{\text{tripod}} \overset{\text{def}}{=} \pi_1^{\text{pro-}l}(Y_{n-1}^{\log})$ (for a suitable choice of basepoint).

(ii) If $(g,r) \neq (1,1)$ or $n \geq 3$, then it holds that V is a tripodal divisor if and only if $Z_{\Pi_n}(I_V)/I_V$ is isomorphic to Π_{n-1}.

(iii) If $(g,r) = (1,1)$ and $n = 2$, then there exist distinct log divisors E, W_1, W_2, W_3 of $X_n^{\log}$ such that

$$\{\textit{log divisors of } X_n^{\log}\} = \{E, W_1, W_2, W_3\},$$

$$\{\textit{tripodal divisors of } X_n^{\log}\} = \{W_1, W_2, W_3\},$$

$$\{\textit{log-full points of } X_n^{\log}\} = \{E \cap W_1, E \cap W_2, E \cap W_3\}.$$

(iv) *If* $(g,r) = (1,1)$ *and* $n = 2$, *then it holds that* V *is a tripodal divisor if and only if there exists a log-full subgroup* A *of* Π_n *such that* A *does not contain any inertia group associated to* V.

Proof. Assertions (i), (ii) follow from Lemma 6.1; Remarks 6.3, 6.4; [Hsh], Corollary 2. Assertion (iii) follows immediately from the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$. Assertion (iv) follows from assertion (iii) and Proposition 4.3. $\square$

Theorem 6.6. *For* $\square \in \{\circ, \bullet\}$, *let* $l^\square$ *be a prime number;* $k^\square$ *an algebraically closed field of characteristic* $\neq l^\square$; $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ *a pair of nonnegative integers such that* $2g^\square - 2 + r^\square > 0$;

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log \square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log \square} \to S^\square$; $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log \square})$ *(for a suitable choice of basepoint);*

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$, *and that* ϕ *induces a bijection between the set of log-full subgroups of* Π° *and the set of log-full subgroups of* $\Pi^\bullet$. *Then* ϕ *induces a bijection between the set of inertia groups of* Π° *associated to tripodal divisors of* $X_{n^\circ}^{\log \circ}$ *and the set of inertia groups of* $\Pi^\bullet$ *associated to tripodal divisors of* $X_{n^\bullet}^{\log \bullet}$.

Proof. Theorem 6.6 follows from Remark 6.3; Theorem 5.2; Lemma 6.5, (ii), (iv). $\square$

7. Reconstruction of drift diagonals

We continue with the notation of §6. In the present §7, we reconstruct the set of inertia groups associated to drift diagonals (cf. Theorem 7.3 below).

Lemma 7.1. *The outer homomorphism* $\iota_\Pi \colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ *induced by* $\iota \colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$ *(cf. Definition 2.2, (viii)) is a surjection whose kernel is topologically generated by the inertia groups associated to the naive diagonals.*

Proof. It follows from [Hsh], Remark B.2, that we have a natural commutative diagram

$$\begin{array}{ccc} \pi_1^{\text{pro-}l}(U_{X_n}) & \longrightarrow & \pi_1^{\text{pro-}l}(U_{X_1}) \times \cdots \times \pi_1^{\text{pro-}l}(U_{X_1}) \\ \downarrow & & \downarrow \\ \Pi_n & \xrightarrow[\iota_\Pi]{} & \Pi_1 \times \cdots \times \Pi_1, \end{array}$$

where $\pi_1^{\text{pro-}l}(U_{X_n}) \to \pi_1^{\text{pro-}l}(U_{X_1}) \times \cdots \times \pi_1^{\text{pro-}l}(U_{X_1})$ denotes the outer surjective homomorphism induced by the open immersion $U_{X_n} \hookrightarrow U_{X_1} \times_S \cdots \times_S U_{X_1}$; the two vertical arrows are isomorphisms. Thus, it follows from the definition of the notion of an inertia group that $\iota_\Pi \colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ is a surjection whose kernel is topologically generated by the inertia groups associated to the naive diagonals. This completes the proof of Lemma 7.1. $\square$

Lemma 7.2. *Let V be a tripodal divisor and I_V an inertia group associated to V. Write $\iota_\Pi\colon \Pi_n \to \Pi_1 \times \cdots \times \Pi_1$ for the outer homomorphism induced by $\iota\colon X_n^{\log} \to X^{\log} \times_S \cdots \times_S X^{\log}$ (cf. Definition 2.2, (viii)). Then the following hold:*

(i) If V is a naive diagonal, then $\iota_\Pi(I_V) = \{1_{\Pi_1\times\cdots\times\Pi_1}\}$.
(ii) If V is not a naive diagonal, then $\iota_\Pi(I_V) \neq \{1_{\Pi_1\times\cdots\times\Pi_1}\}$.

Proof. Assertion (i) follows from Lemma 7.1. Assertion (ii) follows immediately the easily verified fact (i.e., by applying induction on n, together with Proposition 4.1, (i)) that if V is not a naive diagonal, then there exists an $i \in \{1, \ldots, n\}$ such that the projection $p_i\colon X_n^{\log} \to X^{\log}$ maps V to a cusp of $X^{\log}$. $\square$

Theorem 7.3. *For $\square \in \{\circ, \bullet\}$, let $l^\square$ be a prime number; $k^\square$ an algebraically closed field of characteristic $\neq l^\square$; $S^\square \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log\,\square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log\,\square}$ the $n^\square$-th log configuration space associated to $X^{\log\,\square} \to S^\square$; $\Pi^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log\,\square})$ (for a suitable choice of basepoint);

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$, and that ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of inertia groups of Π° associated to drift diagonals of $X_{n^\circ}^{\log\,\circ}$ and the set of inertia groups of $\Pi^\bullet$ associated to drift diagonals of $X_{n^\bullet}^{\log\,\bullet}$.

Proof. First, let us observe that $(g^\circ, r^\circ, n^\circ) = (g^\bullet, r^\bullet, n^\bullet)$ (cf. Theorem 3.10, (i)). Next, let us observe that when $(g^\circ, r^\circ) = (g^\bullet, r^\bullet) = (0, 3)$ or $(1, 1)$, Theorem 7.3 follow formally from Theorem 6.6 and Proposition 3.4, (iii).

Thus, in the remainder of the proof of Theorem 7.3, we suppose that $(g^\circ, r^\circ) = (g^\bullet, r^\bullet) \neq (0, 3), (1, 1)$. Write $\Pi_1^\square \stackrel{\mathrm{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log\,\square})$. Then it follows from Theorem 3.10, (iii), that ϕ induces a commutative diagram

$$\begin{tikzcd}
\Pi^\circ \arrow[rr, "\phi"] \arrow[d, "\iota_\Pi^\circ"'] & & \Pi^\bullet \arrow[d, "\iota_\Pi^\bullet"'] \\
\Pi_1^\circ \times \cdots \times \Pi_1^\circ \arrow[rr, dotted, "\sim"] & & \Pi_1^\bullet \times \cdots \times \Pi_1^\bullet,
\end{tikzcd}$$

where $\iota_\Pi^\square\colon \Pi^\square \to \Pi_1^\square \times \cdots \times \Pi_1^\square$ is the outer homomorphism induced by $\iota^\square\colon X_{n^\square}^{\log\,\square} \to X^{\log\,\square} \times_{S^\square} \cdots \times_{S^\square} X^{\log\,\square}$ (cf. Definition 2.2, (viii)). Thus, the proof of Theorem 7.3 in the case where $(g^\circ, r^\circ) = (g^\bullet, r^\bullet) \neq (0, 3), (1, 1)$ follows formally from Theorem 6.6; Lemma 7.2; Proposition 3.4, (i), (ii). $\square$

8. **Reconstruction of drift collections**

We continue with the notation of §7. In the present §8, we reconstruct the drift collections of Π_n (cf. Theorem 8.14 below).

Definition 8.1. Let Λ be a set of drift diagonals of $X_n^{\log}$. Then we shall say that Λ is a *drift collection* of $X_n^{\log}$ if there exists an automorphism α of $X_n^{\log}$ over S such that $\Lambda = \{\alpha(V) \mid V \text{ is a naive diagonal}\}$.

Definition 8.2. Let V_1, V_2 be distinct drift diagonals and I_{V_1}, I_{V_2} inertia groups associated to V_1, V_2, respectively.

(i) Since V_1, V_2 are tripodal divisors (cf. Proposition 3.4, (i)), and $n > 1$, there exists a unique vertex v_1 (resp. v_2) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$) such that v_1, v_2 are tripods. We shall say that $\{V_1, V_2\}$ is a *scheme-theoretically co-cuspidal pair* if there exists a cusp $y \in C_{r,n}$ which is a cusp of $\mathcal{G}_{V_1}|_{v_1}$, $\mathcal{G}_{V_2}|_{v_2}$.

(ii) We shall say that $\{V_1, V_2\}$ is a *group-theoretically co-cuspidal pair* if there is no log-full subgroup A such that there exist conjugates of I_{V_1}, I_{V_2} that are contained in A.

(iii) We shall say that $\{V_1, V_2\}$ is a *non-intersecting drift pair* if $V_1 \cap V_2 = \emptyset$.

Lemma 8.3. *Let V_1, V_2 be distinct drift diagonals. Then it holds that*

$$\begin{aligned}&\{V_1, V_2\} \textit{ is a group-theoretically co-cuspidal pair}\\ &\iff \textit{there is no log-full point contained in } V_1 \cap V_2.\end{aligned}$$

Proof. This follows immediately from Proposition 4.3. □

Lemma 8.4. *Let V_1, V_2 be log divisors. Then it holds that*

$$V_1 \cap V_2 \neq \emptyset \iff \textit{there is a log-full point contained in } V_1 \cap V_2.$$

Proof. The implication $\Longleftarrow$ is immediate. Thus, it suffices to verify the implication $\Longrightarrow$. Suppose that $V_1 \cap V_2 \neq \emptyset$. Let $P \in V_1 \cap V_2$ be a point and $Q \in X_n^{\log}$ such that $\sharp\mathrm{Node}(\mathcal{G}_Q) = n$ and $\mathcal{G}_P$ is obtained from $\mathcal{G}_Q$ by generization (with respect to some subset of $\mathrm{Node}(\mathcal{G}_Q)$ (cf. [CbTpI], Definition 2.8)). Then it follows from the equivalence (i) $\iff$ (ii) of Proposition 2.9 that $Q \in V_1 \cap V_2$. On the other hand, by Proposition 3.6, it holds that Q is a log-full point. This completes the proof of the implication $\Longrightarrow$. □

Lemma 8.5. *Every scheme-theoretically co-cuspidal pair is group-theoretically co-cuspidal.*

Proof. Let $\{V_1, V_2\}$ be a scheme-theoretically co-cuspidal pair, v_1 the unique vertex of $\mathcal{G}_{V_1}$ which is a tripod, and $y_1, y_2 \in C_{r,n}$ the two cusps of $\mathcal{G}_{V_1}|_{v_1}$. By Lemma 8.3, to complete the proof of Lemma 8.5, it suffices to derive a contradiction under the assumption that $V_1 \cap V_2$ contains a log-full point P. Thus, suppose that this assumption holds. Then v_1 determines a unique vertex v_1^P of $\mathcal{G}_P$, which is necessarily a tripod (cf. [CbTpI], Definition 2.8, (iii)). In particular, since $\mathcal{G}_{V_2}$ may be regarded as a generization of $\mathcal{G}_P$, the vertex v_1^P of $\mathcal{G}_P$ determines a vertex w_2 of $\mathcal{G}_{V_2}$ such that y_1, y_2 are cusps of $\mathcal{G}_{V_2}|_{w_2}$ (cf. [CbTpI], Definition 2.8, (iii)). Since $\{V_1, V_2\}$ is a scheme-theoretically co-cuspidal pair, it thus follows from Remark 2.4 that $V_1 = V_2$, a contradiction. □

Lemma 8.6. *Every non-intersecting drift pair is scheme-theoretically co-cuspidal.*

Proof. Let $\{V_1, V_2\}$ be a pair of distinct drift diagonals which is not a scheme-theoretically co-cuspidal pair. Then since $n > 1$, there exists a unique vertex v_1 (resp. v_2) of $\mathcal{G}_{V_1}$ (resp. $\mathcal{G}_{V_2}$) such that v_1, v_2 are tripods (cf. Proposition 3.4, (i)). Since $\{V_1, V_2\}$ is not a scheme-theoretically co-cuspidal pair, there exist cusps

y_1, z_1 of $\mathcal{G}_{V_1}|_{v_1}$ and cusps y_2, z_2 of $\mathcal{G}_{V_2}|_{v_2}$ such that $y_1, z_1, y_2, z_2 \in C_{r,n}$ are distinct elements, $\sharp(\{y_1, z_1\} \cap \{c_1, \ldots, c_r\}) \leq 1$, $\sharp(\{y_2, z_2\} \cap \{c_1, \ldots, c_r\}) \leq 1$ (cf. Definition 2.3). Thus, it follows from the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ that there exist a point P of $X_n^{\log}$ and terminal vertices t_1, t_2 of $\mathcal{G}_P$ such that t_1, t_2 are tripods, y_1, z_1 are cusps of $\mathcal{G}_P|_{t_1}$, and y_2, z_2 are cusps of $\mathcal{G}_P|_{t_2}$. In particular, by the equivalence (i) $\iff$ (iii) of Proposition 2.9, it holds that $P \in V_1 \cap V_2$. Thus, $\{V_1, V_2\}$ is not a non-intersecting drift pair. $\square$

Proposition 8.7. *Let V_1, V_2 be distinct drift diagonals. Then it holds that*

$$\begin{aligned} &\{V_1, V_2\} \text{ is a scheme-theoretically co-cuspidal pair} \\ \iff &\{V_1, V_2\} \text{ is a group-theoretically co-cuspidal pair} \\ \iff &\{V_1, V_2\} \text{ is a non-intersecting drift pair.} \end{aligned}$$

Proof. This follows immediately from Lemmas 8.3, 8.4, 8.5, 8.6. $\square$

Definition 8.8. Let V_1, V_2, V_3 be distinct drift diagonals. Then we shall say that $\{V_1, V_2, V_3\}$ is a *scheme-theoretically co-cuspidal triple* if $\{V_1, V_2\}$, $\{V_2, V_3\}$, and $\{V_3, V_1\}$ are scheme-theoretically co-cuspidal pairs.

Definition 8.9. Let Λ be a set of drift diagonals such that $\sharp\Lambda = \frac{n(n-1)}{2}$. We shall say that Λ is a *scheme-theoretic drift collection* of $X_n^{\log}$ if there exist distinct drift diagonals $V_{i,j}$, where $i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}$, such that $\Lambda = \{V_{i,j} \mid i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}\}$, and, moreover, the following hold:

(a) For any $i \in \{1, \ldots, n-2\}$, $\{V_{i,i+1}, V_{i+1,i+2}\}$ is a scheme-theoretically co-cuspidal pair.
(b) For any $i \in \{1, \ldots, n-2\}, j \in \{i+2, \ldots, n-1\}$, then $\{V_{i,i+1}, V_{j,j+1}\}$ is not a scheme-theoretically co-cuspidal pair.
(c) For any $i \in \{1, \ldots, n-2\}, j \in \{i+2, \ldots, n\}$, $\{V_{i,j}, V_{i,i+1}, V_{i+1,j}\}$ is a scheme-theoretically co-cuspidal triple.

Lemma 8.10. *Every drift collection of $X_n^{\log}$ is a scheme-theoretic drift collection of $X_n^{\log}$.*

Proof. Let Λ be a drift collection of $X_n^{\log}$. Then it follows from Proposition 3.3, (i) (cf. also Remark 3.2), that there exist distinct elements $y_1, \ldots, y_n \in C_{r,n}$ such that

$$\Lambda = \{V(\{y_i, y_j\}) \mid i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}.$$

Then one verifies easily that if we write $V_{i,j} \stackrel{\text{def}}{=} V(y_i, y_j)$, then the $V_{i,j}$'s satisfy the conditions of Definition 8.9, and hence that Λ is a scheme-theoretic drift collection of $X_n^{\log}$. $\square$

Lemma 8.11. *Every scheme-theoretic drift collection of $X_n^{\log}$ is a drift collection of $X_n^{\log}$.*

Proof. By Proposition 3.4, (ii), we may assume without loss of generality that $(g, r) = (0, 3)$ or $(1, 1)$. Let Λ be a scheme-theoretic drift collection of $X_n^{\log}$. By Definitions 2.3; 8.2, (i); 8.9, (a), there exist elements $y_1, y_2, y_3 \in C_{r,n}$ such that $V_{1,2} = V(\{y_1, y_2\}), V_{2,3} = V(\{y_2, y_3\})$. By Definitions 2.3; 8.2, (i); 8.9, (a), (b), there exist elements $y_4, \ldots, y_n \in C_{r,n}$ such that $V_{i,i+1} = V(\{y_i, y_{i+1}\})$. Thus, by Definitions 2.3; 8.8; 8.9, (c), it holds that $V_{i,j} = V(\{y_i, y_j\})$. Finally, since

$(g, r) = (0, 3)$ or $(1, 1)$, by applying a suitable automorphism of $X_n^{\log}$ that arises from a permutation of the $r+n$ marked points of the stable log curve $X_{n+1}^{\log} \to X_n^{\log}$, it follows that $\Lambda = \{V(\{y_i, y_j\}) \mid i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}\}$ is a drift collection of $X_n^{\log}$. $\square$

Proposition 8.12. *Let Λ be a set of drift diagonals of $X_n^{\log}$. Then Λ is a drift collection of $X_n^{\log}$ if and only if Λ is a scheme-theoretic drift collection of $X_n^{\log}$.*

Proof. This follows immediately from Lemmas 8.10, 8.11. $\square$

Definition 8.13. We shall refer to as a *drift collection* of Π_n any collection

$$\{I_V \mid V \in \Lambda\}$$

of subgroups of Π_n associated to some drift collection Λ of $X_n^{\log}$, where I_V denotes an inertia group of Π_n associated to $V \in \Lambda$.

Theorem 8.14. *For $\square \in \{\circ, \bullet\}$, let $l^\square$ be a prime number; $k^\square$ an algebraically closed field of characteristic $\neq l^\square$; $S^\square \stackrel{\text{def}}{=} \operatorname{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$; $n^\square \in \mathbb{Z}_{>1}$; $X_{n^\square}^{\log \square}$ the $n^\square$-th log configuration space associated to $X^{\log \square} \to S^\square$; $\Pi^\square \stackrel{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X_{n^\square}^{\log \square})$ (for a suitable choice of basepoint);

$$\phi \colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$, and that ϕ induces a bijection between the set of log-full subgroups of Π° and the set of log-full subgroups of $\Pi^\bullet$. Then ϕ induces a bijection between the set of drift collections of Π° and the set of drift collections of $\Pi^\bullet$ (cf. Definition 8.13).

Proof. This follows from Theorem 7.3 and Propositions 8.7, 8.12. $\square$

9. Reconstruction of generalized fiber subgroups

We continue with the notation of §8. In the present §9, we reconstruct the generalized fiber subgroups of Π_n (cf. Theorem 9.3 below).

Definition 9.1. Let H be a closed subgroup of Π_n. We shall say that H is a *generalized fiber subgroup* if there exist an automorphism α of $X_n^{\log}$ over S and a fiber subgroup $F \subseteq \Pi_n$ (cf. [MzTa], Definition 2.3, (iii)) such that $H = \beta(F)$, where β is an automorphism of Π_n which arises from α (cf. Remark 3.2; [HMM], Definition 2.1, (ii); [HMM], Remark 2.1.1).

Proposition 9.2. *If $(g, r) \neq (0, 3), (1, 1)$, then*

$$\{\textit{generalized fiber subgroups}\} = \{\textit{fiber subgroups}\}.$$

Proof. This follows immediately from Remark 3.2. $\square$

Theorem 9.3. *For $\square \in \{\circ, \bullet\}$, let $l^\square$ be a prime number; $k^\square$ an algebraically closed field of characteristic $\neq l^\square$; $S^\square \stackrel{\text{def}}{=} \operatorname{Spec}(k^\square)$; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$;*

$$X^{\log \square} \to S^\square$$

a smooth log curve of type $(g^\square, r^\square)$*;* $n^\square \in \mathbb{Z}_{>1}$*;* $X^{\log\square}_{n^\square}$ *the* $n^\square$*-th log configuration space associated to* $X^{\log\square} \to S^\square$*;* $\Pi^\square \overset{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log\square}_{n^\square})$ *(for a suitable choice of basepoint);*

$$\phi\colon \Pi^\circ \xrightarrow{\sim} \Pi^\bullet$$

an isomorphism of profinite groups. We suppose that $r^\square > 0$*, and that* ϕ *induces a bijection between the set of log-full subgroups of* Π° *and the set of log-full subgroups of* $\Pi^\bullet$*. Then* ϕ *induces a bijection between the set of generalized fiber subgroups of* Π° *and the set of generalized fiber subgroups of* $\Pi^\bullet$ *(cf. Definition 9.1).*

Proof. Write $\Pi_1^\square \overset{\text{def}}{=} \pi_1^{\text{pro-}l^\square}(X^{\log\square})$. For any drift collection $\Lambda^\square$ of $\Pi^\square$, write $\iota^\square\colon \Pi^\square \to Q^\square_{\Lambda^\square}$ for the surjection obtained by forming the quotient by the normal closed subgroup generated by the subgroups $\subseteq \Pi^\square$ that constitute the drift collection $\Lambda^\square$. Recall that it follows from Lemma 7.1 that there exist $n^\square$ surjections $Q^\square_{\Lambda^\square} \to \Pi_1^\square$, which we shall refer to as $\Lambda^\square$-*projections*, such that the resulting product homomorphism determines an isomorphism $Q^\square_{\Lambda^\square} \xrightarrow{\sim} \Pi_1^\square \times \cdots \times \Pi_1^\square$.

Let $F^\circ \subseteq \Pi^\circ$ be a generalized fiber subgroup of Π°. Then one verifies immediately that there exists a drift collection Λ° of Π° such that F° is contained in the kernel $\mathrm{Ker}(p^\circ)$ of some Λ°-projection p°. Write $\Lambda^\bullet$ for the drift collection of $\Pi^\bullet$ determined by applying ϕ to Λ° (cf. Theorem 8.14).

Next, observe that since each factor "Π_1" of the n factors of the product "$\Pi_1 \times \cdots \times \Pi_1$" of Lemma 7.1 is *slim* (cf., e.g., [MzTa], Proposition 1.4), it follows that each such factor "Π_1" may be reconstructed as the *centralizer* of any product of open subgroups of the remaining $n-1$ factors. In particular, it follows immediately from [MzTa], Corollary 3.4, that there exists a commutative diagram

$$\begin{tikzcd} \Pi^\circ \arrow[r, "\phi"] \arrow[d, "p^\circ"'] & \Pi^\bullet \arrow[d, "p^\bullet"'] \\ \Pi_1^\circ \arrow[r, dotted] & \Pi_1^\bullet, \end{tikzcd}$$

where $p^\bullet$ is a $\Lambda^\bullet$-projection, and the horizontal arrows are isomorphisms. On the other hand, it follows immediately from the definition of $p^\square$ that $\mathrm{Ker}(p^\square)$ has a natural structure of configuration space group (whose "(g, r)" is $\neq (0,3), (1,1)$!), and that F° is a fiber subgroup of $\mathrm{Ker}(p^\circ)$ (cf. [MzTa], Proposition 2.4, (i), (ii)). Thus, by [MzTa], Corollary 6.3, $F^\bullet \overset{\text{def}}{=} \phi(F^\circ)$ is a fiber subgroup of $\mathrm{Ker}(p^\bullet)$, hence also of $\Pi^\bullet$. $\square$

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